

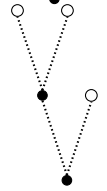


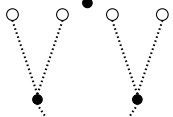
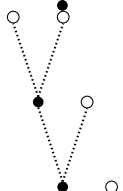
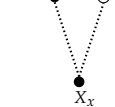
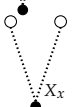


## Renormalized equation

$$(\partial_t - \Delta) u = u_x^2 + \xi$$

with  $\xi \in C^{-3/2-\kappa}$ , parabolic scaling  $s = (2, 1)$ , spatial dimension  $d = 1$ .

### Divergent trees

$\tau$	$ \tau $	$S(\tau)$	$k_\tau$	$F(\tau^*)$
	$-3/2 - \kappa$	1	$k_0$	1
	$-1 - 2\kappa$	2	$k_2$	2
	$-1/2 - 3\kappa$	2	$k_3$	4
	$-1/2 - \kappa$	1	-	0
	$-1/2 - \kappa$	1	$k_1$	$2u_x$
	$-4\kappa$	8	$k_6$	8
	$-4\kappa$	2	$k_7$	8
	$-2\kappa$	1	-	0
	$-2\kappa$	2	-	0

$\tau$	$ \tau $	$S(\tau)$	$k_\tau$	$F(\tau^*)$
	$-2\kappa$	1	$k_4$	$4u_x$
	$-2\kappa$	2	$k_5$	$4u_x$

Legend:  $\circ$  = noise,  $\bullet$  = integration node, solid edge =  $\mathcal{I}$ , dotted edge =  $\partial\mathcal{I}$  (derivative),  $X^n$  above a node = polynomial (Taylor) decoration, e.g.  $X_x$ ; recall  $(\Pi X^n)(y) = y^n$ .

## Renormalized family

$$\begin{aligned}
 (\partial_t - \Delta) u &= u_x^2 + \xi \\
 &+ k_0 \\
 &+ k_2 \\
 &+ 2k_3 \\
 &+ 2k_1 u_x \\
 &+ k_6 \\
 &+ 4k_7 \\
 &+ 4k_4 u_x \\
 &+ 2k_5 u_x
 \end{aligned}$$

## Canonical (BPHZ) renormalization

Each free constant at its canonical value  $k_\tau = h(S'_-\tau)$  for a centered Gaussian noise. Mean-zero parity makes trees with an odd number of noise vertices vanish (3 of 8 below); the survivors are polynomials in the elementary expectations  $h_i \equiv h_\varepsilon(\sigma) = \mathbb{E}[\Pi^\varepsilon \sigma](0)$  of the  $\varepsilon$ -regularized noise  $\xi_\varepsilon$ .

These constants are  $\varepsilon$ -dependent and diverge as  $\varepsilon \rightarrow 0$ ; the integrals below are left unevaluated.

$$k_0 = 0 \quad (\text{vanishes: odd noise parity})$$

$$k_2 = -h_0$$

$$k_3 = 0 \quad (\text{vanishes: odd noise parity})$$

$$k_1 = 0 \quad (\text{vanishes: odd noise parity})$$

$$k_6 = 2h_0^2 h_2 h_8 - h_0^2 h_7 - 2h_0 h_1 h_8 - 2h_0 h_2 h_6 + 2h_0 h_5 + 2h_1 h_6 - h_4$$

$$k_7 = h_0 h_{10} - h_0 h_{11} h_2 - h_0 h_3 h_8 + h_1 h_{11} + 2h_{11} h_3 + h_3 h_6 - h_9$$

$$k_4 = -h_3$$

$$k_5 = h_0 h_2 - h_1$$

giving the canonically renormalized equation(s)

$$\begin{aligned} (\partial_t - \Delta) u &= u_x^2 + \xi \\ &+ -h_0 \\ &+ 2h_0^2 h_2 h_8 - h_0^2 h_7 - 2h_0 h_1 h_8 - 2h_0 h_2 h_6 + 2h_0 h_5 + 2h_1 h_6 - h_4 \\ &+ 4h_0 h_{10} - 4h_0 h_{11} h_2 - 4h_0 h_3 h_8 + 4h_1 h_{11} + 8h_{11} h_3 + 4h_3 h_6 - 4h_9 \\ &+ -4h_3 u_x \\ &+ 2h_0 h_2 u_x - 2h_1 u_x \end{aligned}$$

where, for the  $\varepsilon$ -regularized noise  $\xi_\varepsilon$  (covariance  $C_\varepsilon$ ) and the singular kernel  $K$  of  $L^{-1}$ , the elementary expectations are given below. An entry marked = 0 vanishes, and ( $= h_j$ ) marks a duplicate (same value up to the  $o$ -decoration).

$$h_0 = h_\varepsilon \left( \begin{array}{c} \circ \quad \circ \\ \diagdown \quad \diagup \\ \bullet \end{array} \right)$$

$$h_0 = \int_{(\mathbb{R}^2)^2} C_\varepsilon(z_1 - z_2) \partial^{(0,1)} K(-z_1) \partial^{(0,1)} K(-z_2) dz_1 dz_2$$

$$h_1 = h_\varepsilon \left( \begin{array}{c} \circ \quad \circ \\ \diagdown \quad \diagup \\ \bullet \\ \vdots \\ \bullet \end{array} \right)$$

$$h_1 = \int_{(\mathbb{R}^2)^3} C_\varepsilon(z_2 - z_3) \partial^{(0,1)} K(-z_1) \partial^{(0,1)} K(z_1 - z_2) \partial^{(0,1)} K(z_1 - z_3) dz_1 dz_2 dz_3$$

$$h_2 = h_\varepsilon \left( \begin{array}{c} o = -1 - 2\kappa \\ \square \\ \vdots \\ \bullet \end{array} \right)$$

$$h_2 = \int_{(\mathbb{R}^2)^1} \partial^{(0,1)} K(-z_1) dz_1$$

= 0 — pure-kernel total derivative

$$h_3 = h_\varepsilon \left( \begin{array}{c} \circ \\ \vdots \\ \bullet \quad \circ \\ \vdots \\ \bullet \end{array} \right)$$

$$h_3 = \int_{(\mathbb{R}^2)^3} C_\varepsilon(z_2 - z_3) \partial^{(0,1)} K(-z_1) \partial^{(0,1)} K(-z_3) \partial^{(0,1)} K(z_1 - z_2) dz_1 dz_2 dz_3$$

$$h_4 = h_\varepsilon \left( \begin{array}{c} \circ \quad \circ \\ \vdots \quad \vdots \\ \bullet \quad \bullet \\ \vdots \\ \bullet \end{array} \right)$$

$$h_4 = \int_{(\mathbb{R}^2)^6} C_\varepsilon(z_2 - z_3) C_\varepsilon(z_5 - z_6) \partial^{(0,1)} K(-z_1) \partial^{(0,1)} K(-z_4) \\ \partial^{(0,1)} K(z_1 - z_2) \partial^{(0,1)} K(z_1 - z_3) \partial^{(0,1)} K(z_4 - z_5) \partial^{(0,1)} K(z_4 - z_6) dz_1 dz_2 dz_3 dz_4 dz_5 dz_6 \\ + \int_{(\mathbb{R}^2)^6} C_\varepsilon(z_2 - z_5) C_\varepsilon(z_3 - z_6) \partial^{(0,1)} K(-z_1) \partial^{(0,1)} K(-z_4) \\ \partial^{(0,1)} K(z_1 - z_2) \partial^{(0,1)} K(z_1 - z_3) \partial^{(0,1)} K(z_4 - z_5) \partial^{(0,1)} K(z_4 - z_6) dz_1 dz_2 dz_3 dz_4 dz_5 dz_6 \\ + \int_{(\mathbb{R}^2)^6} C_\varepsilon(z_2 - z_6) C_\varepsilon(z_3 - z_5) \partial^{(0,1)} K(-z_1) \partial^{(0,1)} K(-z_4) \\ \partial^{(0,1)} K(z_1 - z_2) \partial^{(0,1)} K(z_1 - z_3) \partial^{(0,1)} K(z_4 - z_5) \partial^{(0,1)} K(z_4 - z_6) dz_1 dz_2 dz_3 dz_4 dz_5 dz_6$$

$$h_5 = h_\varepsilon \left( \begin{array}{c} \circ \quad \circ \\ \vdots \quad \vdots \\ \bullet \quad \square \\ \vdots \\ \bullet \end{array} \right) \quad \begin{array}{l} o = -1 - 2\kappa \\ \square \end{array}$$

$$h_5 = \int_{(\mathbb{R}^2)^4} C_\varepsilon(z_2 - z_3) \partial^{(0,1)} K(-z_1) \partial^{(0,1)} K(-z_4) \partial^{(0,1)} K(z_1 - z_2) \\ \partial^{(0,1)} K(z_1 - z_3) dz_1 dz_2 dz_3 dz_4$$

$$h_6 = h_\varepsilon \left( \begin{array}{c} \circ \quad \circ \\ \vdots \quad \vdots \\ \bullet \\ \vdots \\ \square \end{array} \right) \quad \begin{array}{l} (o = -2\kappa) \\ \square \\ o = 2\kappa \end{array}$$

$$h_6 = \int_{(\mathbb{R}^2)^3} C_\varepsilon(z_2 - z_3) \partial^{(0,1)}K(-z_1) \partial^{(0,1)}K(z_1 - z_2) \partial^{(0,1)}K(z_1 - z_3) dz_1 dz_2 dz_3$$

$$= h_1 \text{ (same value)}$$

$$h_7 = h_\varepsilon \left( \begin{array}{c} o = -1 \quad o = 2\kappa + 1 - 2\kappa \\ \square \quad \square \\ \diagdown \quad \diagup \\ \bullet \end{array} \right)$$

$$h_7 = \int_{(\mathbb{R}^2)^2} \partial^{(0,1)}K(-z_1) \partial^{(0,1)}K(-z_2) dz_1 dz_2$$

$$= 0 \text{ — pure-kernel total derivative}$$

$$h_8 = h_\varepsilon \left( \begin{array}{c} o = -1 - 2\kappa \\ \vdots \\ \square \\ \vdots \\ \square \\ o = 2\kappa \end{array} \right) \quad (o = -2\kappa)$$

$$h_8 = \int_{(\mathbb{R}^2)^1} \partial^{(0,1)}K(-z_1) dz_1$$

$$= 0 \text{ — pure-kernel total derivative}$$

$$h_9 = h_\varepsilon \left( \begin{array}{c} \circ \quad \circ \\ \diagdown \quad \diagup \\ \bullet \\ \diagdown \quad \diagup \\ \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} \right)$$

$$\begin{aligned} h_9 = & \int_{(\mathbb{R}^2)^6} C_\varepsilon(z_3 - z_4) C_\varepsilon(z_5 - z_6) \partial^{(0,1)}K(-z_1) \partial^{(0,1)}K(-z_6) \\ & \partial^{(0,1)}K(z_1 - z_2) \partial^{(0,1)}K(z_1 - z_5) \partial^{(0,1)}K(z_2 - z_3) \partial^{(0,1)}K(z_2 - z_4) dz_1 dz_2 dz_3 dz_4 dz_5 dz_6 \\ & + \int_{(\mathbb{R}^2)^6} C_\varepsilon(z_3 - z_5) C_\varepsilon(z_4 - z_6) \partial^{(0,1)}K(-z_1) \partial^{(0,1)}K(-z_6) \\ & \partial^{(0,1)}K(z_1 - z_2) \partial^{(0,1)}K(z_1 - z_5) \partial^{(0,1)}K(z_2 - z_3) \partial^{(0,1)}K(z_2 - z_4) dz_1 dz_2 dz_3 dz_4 dz_5 dz_6 \\ & + \int_{(\mathbb{R}^2)^6} C_\varepsilon(z_3 - z_6) C_\varepsilon(z_4 - z_5) \partial^{(0,1)}K(-z_1) \partial^{(0,1)}K(-z_6) \\ & \partial^{(0,1)}K(z_1 - z_2) \partial^{(0,1)}K(z_1 - z_5) \partial^{(0,1)}K(z_2 - z_3) \partial^{(0,1)}K(z_2 - z_4) dz_1 dz_2 dz_3 dz_4 dz_5 dz_6 \end{aligned}$$

$$h_{10} = h_\varepsilon \left( \begin{array}{c} o = -1 - 2\kappa \\ \square \quad \circ \\ \diagdown \quad \diagup \\ \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} \right)$$

$$h_{10} = \int_{(\mathbb{R}^2)^4} C_\varepsilon(z_3 - z_4) \partial^{(0,1)} K(-z_1) \partial^{(0,1)} K(-z_4) \partial^{(0,1)} K(z_1 - z_2) \partial^{(0,1)} K(z_1 - z_3) dz_1 dz_2 dz_3 dz_4$$

$$h_{11} = h_\varepsilon \left( \begin{array}{c} \circ \\ \vdots \\ \square \\ \vdots \\ \bullet \end{array} \begin{array}{c} \\ \\ \circ \end{array} \right)$$

$$h_{11} = \int_{(\mathbb{R}^2)^3} C_\varepsilon(z_2 - z_3) \partial^{(0,1)} K(-z_1) \partial^{(0,1)} K(-z_3) \partial^{(0,1)} K(z_1 - z_2) dz_1 dz_2 dz_3$$

=  $h_3$  (same value)