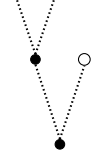
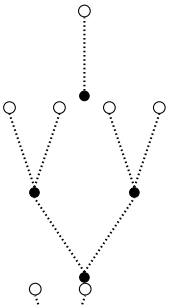
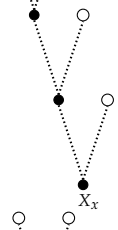
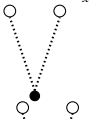
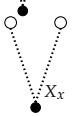
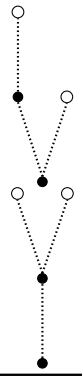
Renormalized equation

$$(\partial_t - \Delta)\, u = u_x^2 + \xi$$

with $\xi \in C^{-3/2-\kappa}$, parabolic scaling $\mathfrak{s} = (2, 1)$, spatial dimension $d = 1$.

Divergent trees

τ	$\lvert\tau\rvert$	$S(\tau)$	k_τ	$F(\tau^*)$
	$-3/2-\kappa$	1	k_0	1
	$-1-2\kappa$	2	k_2	2
	$-1/2-3\kappa$	2	k_3	4
	$-1/2-\kappa$	1	–	0
	$-1/2-\kappa$	1	k_1	$2u_x$
	-4κ	8	k_6	8
	-4κ	2	k_7	8
	-2κ	1	–	0
	-2κ	2	–	0

τ	$\lvert\tau\rvert$	$S(\tau)$	k_τ	$F(\tau^*)$
	-2κ	1	k_4	$4u_x$
	-2κ	2	k_5	$4u_x$

Legend: ○ = noise, ● = integration node, solid edge = $\mathcal{I}$, dotted edge = $\partial\mathcal{I}$ (derivative), X^n above a node = polynomial (Taylor) decoration, e.g. X_x; recall $(\Pi X^n)(y) = y^n$.

Renormalized family

$$
\begin{aligned}
(\partial_t - \Delta)\, u = {} & u_x^2 + \xi \\
& + k_0 \\
& + k_2 \\
& + 2k_3 \\
& + 2k_1 u_x \\
& + k_6 \\
& + 4k_7 \\
& + 4k_4 u_x \\
& + 2k_5 u_x
\end{aligned}
$$

Canonical (BPHZ) renormalization — reduced

Each free constant at its canonical value $k_\tau = h(S'_-\tau)$ for a centered Gaussian noise, **reduced for a spatially-symmetric (e.g. white) noise**: provably-zero $h(\sigma)$ set to 0 — root X^n, pure-kernel total derivatives, and odd spatial-reflection parity ($x \to -x$) — and o-duplicate $h(\sigma)$ merged (5

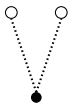
of 8 vanish). Each is an exact identity.

$$
\begin{aligned}
k_0 &= 0 \\
k_2 &= -h_0 \\
k_3 &= 0 \\
k_1 &= 0 \\
k_6 &= 2h_0h_2 - h_3 \\
k_7 &= h_0h_1 - h_4 \\
k_4 &= 0 \\
k_5 &= 0
\end{aligned}
$$

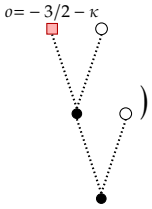
giving the canonically renormalized equation(s)

$$
\begin{aligned}
(\partial_t - \Delta)\, u = u_x^2 &+ \xi \\
&+ -h_0 \\
&+ 2h_0h_2 - h_3 \\
&+ 4h_0h_1 - 4h_4
\end{aligned}
$$

where, for the ε-regularized noise ξ_ε (covariance C_ε) and the singular kernel K of L^{-1}, the elementary expectations are given below.

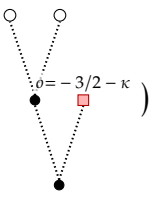
$h_0 = h_\varepsilon($  $)$

$$
h_0 = \int_{(\mathbb{R}^2)^2} C_\varepsilon(z_1 - z_2)\, \partial^{(0,1)}K(-z_1)\, \partial^{(0,1)}K(-z_2)\, \mathrm{d}z_1\, \mathrm{d}z_2
$$

$h_1 = h_\varepsilon($ 

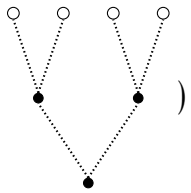
$)$

$$
h_1 = \int_{(\mathbb{R}^2)^4} C_\varepsilon(z_3 - z_4)\, \partial^{(0,1)}K(-z_1)\, \partial^{(0,1)}K(-z_4)\, \partial^{(0,1)}K(z_1 - z_2)\, \partial^{(0,1)}K(z_1 - z_3)\, \mathrm{d}z_1\, \mathrm{d}z_2\, \mathrm{d}z_3\, \mathrm{d}z_4
$$

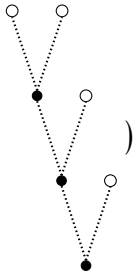
$h_2 = h_\varepsilon($ 

$)$

$$h_2 = \int_{(\mathbb{R}^2)^4} C_\varepsilon(z_2 - z_3)\, \partial^{(0,1)} K(-z_1)\, \partial^{(0,1)} K(-z_4)\, \partial^{(0,1)} K(z_1 - z_2)$$
$$\partial^{(0,1)} K(z_1 - z_3)\ \mathrm{d}z_1\, \mathrm{d}z_2\, \mathrm{d}z_3\, \mathrm{d}z_4$$

$h_3 = h_\varepsilon($  $)$

$$h_3 = \int_{(\mathbb{R}^2)^6} C_\varepsilon(z_2 - z_3)\, C_\varepsilon(z_5 - z_6)\, \partial^{(0,1)} K(-z_1)\, \partial^{(0,1)} K(-z_4)$$
$$\partial^{(0,1)} K(z_1 - z_2)\, \partial^{(0,1)} K(z_1 - z_3)\, \partial^{(0,1)} K(z_4 - z_5)\, \partial^{(0,1)} K(z_4 - z_6)\ \mathrm{d}z_1\, \mathrm{d}z_2\, \mathrm{d}z_3\, \mathrm{d}z_4\, \mathrm{d}z_5\, \mathrm{d}z_6$$
$$+ \int_{(\mathbb{R}^2)^6} C_\varepsilon(z_2 - z_5)\, C_\varepsilon(z_3 - z_6)\, \partial^{(0,1)} K(-z_1)\, \partial^{(0,1)} K(-z_4)$$
$$\partial^{(0,1)} K(z_1 - z_2)\, \partial^{(0,1)} K(z_1 - z_3)\, \partial^{(0,1)} K(z_4 - z_5)\, \partial^{(0,1)} K(z_4 - z_6)\ \mathrm{d}z_1\, \mathrm{d}z_2\, \mathrm{d}z_3\, \mathrm{d}z_4\, \mathrm{d}z_5\, \mathrm{d}z_6$$
$$+ \int_{(\mathbb{R}^2)^6} C_\varepsilon(z_2 - z_6)\, C_\varepsilon(z_3 - z_5)\, \partial^{(0,1)} K(-z_1)\, \partial^{(0,1)} K(-z_4)$$
$$\partial^{(0,1)} K(z_1 - z_2)\, \partial^{(0,1)} K(z_1 - z_3)\, \partial^{(0,1)} K(z_4 - z_5)\, \partial^{(0,1)} K(z_4 - z_6)\ \mathrm{d}z_1\, \mathrm{d}z_2\, \mathrm{d}z_3\, \mathrm{d}z_4\, \mathrm{d}z_5\, \mathrm{d}z_6$$

$h_4 = h_\varepsilon($  $)$

$$h_4 = \int_{(\mathbb{R}^2)^6} C_\varepsilon(z_3 - z_4)\, C_\varepsilon(z_5 - z_6)\, \partial^{(0,1)} K(-z_1)\, \partial^{(0,1)} K(-z_6)$$
$$\partial^{(0,1)} K(z_1 - z_2)\, \partial^{(0,1)} K(z_1 - z_5)\, \partial^{(0,1)} K(z_2 - z_3)\, \partial^{(0,1)} K(z_2 - z_4)\ \mathrm{d}z_1\, \mathrm{d}z_2\, \mathrm{d}z_3\, \mathrm{d}z_4\, \mathrm{d}z_5\, \mathrm{d}z_6$$
$$+ \int_{(\mathbb{R}^2)^6} C_\varepsilon(z_3 - z_5)\, C_\varepsilon(z_4 - z_6)\, \partial^{(0,1)} K(-z_1)\, \partial^{(0,1)} K(-z_6)$$
$$\partial^{(0,1)} K(z_1 - z_2)\, \partial^{(0,1)} K(z_1 - z_5)\, \partial^{(0,1)} K(z_2 - z_3)\, \partial^{(0,1)} K(z_2 - z_4)\ \mathrm{d}z_1\, \mathrm{d}z_2\, \mathrm{d}z_3\, \mathrm{d}z_4\, \mathrm{d}z_5\, \mathrm{d}z_6$$
$$+ \int_{(\mathbb{R}^2)^6} C_\varepsilon(z_3 - z_6)\, C_\varepsilon(z_4 - z_5)\, \partial^{(0,1)} K(-z_1)\, \partial^{(0,1)} K(-z_6)$$
$$\partial^{(0,1)} K(z_1 - z_2)\, \partial^{(0,1)} K(z_1 - z_5)\, \partial^{(0,1)} K(z_2 - z_3)\, \partial^{(0,1)} K(z_2 - z_4)\ \mathrm{d}z_1\, \mathrm{d}z_2\, \mathrm{d}z_3\, \mathrm{d}z_4\, \mathrm{d}z_5\, \mathrm{d}z_6$$