Lecture 11. Risk measures

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Leargning objectives

- Introduce risk measures informally and formally
- Discuss standard examples like VaR, TVaR/ES, distortion risk measures

Risk measures

> **“Definition” 1 (Risk measure).**
> *A risk measure is a function $\rho : \mathbb{R} \to \mathbb{R}$ that assigns a real number to a random variable X* ***interpreted as a loss to quantify how risky it is****.*

The quotation marks around the word “Definition” are there to indicate that the definition is not a formal one, but rather a **conceptual** one. So far it’s not suitable for a mathematical treatment, we need to formalize what we mean by “to quantify how risky it is”.

One natural requirement is that a risk measure should be **monotone**

$$X \leq Y \implies \rho(X) \leq \rho(Y),$$

but what else?

Coherent risk measures

One possible formalization of the concept of risk measure is via **coherent risk measures**. A risk measure ρ is called coherent if it satisfies the following properties for all random variables X, Y and $a > 0$:

- **Monotonicity:** If $X \leq Y$, then $\rho(X) \leq \rho(Y)$ (the risk of a larger loss is larger)
- **Subadditivity:** $\rho(X+Y) \leq \rho(X) + \rho(Y)$ (diversification is beneficial)
- **Positive homogeneity:** $\rho(aX) = a\rho(X)$ for all $a \geq 0$. (no liquidity effects)
- **Translation invariance:** $\rho(X+b) = \rho(X) + b$ for all $b \in \mathbb{R}$ (adding a sure loss increases the risk by that amount)

Interpretation[1]**:** define acceptance set $A_\rho = \{X : \rho(X) \leq 0\}$. If ρ is coherent, then

$$\rho(X) = \inf\{m \in \mathbb{R} : X + m \in A_\rho\} = \begin{pmatrix} \text{minimal cash } m \text{ you must add} \\ \text{to make the position acceptable} \end{pmatrix}.$$

[1] There's also another interpretation (ρ as worst expected loss under plausible scenarios)

Value-at-Risk (VaR)

Definition 1 (Value-at-Risk).
For a given confidence level $\alpha \in (0,1)$, the Value-at-Risk (VaR) of a random variable X is defined as

$$\mathrm{VaR}_X(p) = F^{-1}(p).$$

Alternatively,

$$\mathrm{VaR}_X(p) = \mathrm{argmin}_{y \in \mathbb{R}} \mathbb{E}\left\{S_X(p,y)\right\},$$

where $S_x(p,y)$ is the so-called pinball function

$$S_x(p,y) = p(x-y)^+ + (1-p)(y-x)^+.$$

VaR and monotone transformations

Theorem 2.
If g is non-decreasing, then $\mathrm{VaR}_{g(X)}(p) = g(\mathrm{VaR}_X(p))$.

Important example: If $a > 0$, then $\mathrm{VaR}_{aX+b}(p) = a\,\mathrm{VaR}_X(p) + b$.

Hence, VaR is **positively homogeneous**, **translation-invariant** and **monotone**, but it is not **subadditive**! Therefore, VaR is **not** a coherent risk measure.

Tail Value-at-Risk and Expected shortfall

Let X be a random variable with finite expectation and $p \in (0, 1)$.

Definition 3 (Tail Value-at-Risk).

$$\mathrm{TVaR}_X(p) = \mathbb{E}\left\{X \mid X \geq \mathrm{VaR}_X(p)\right\}.$$

Definition 4 (Expected shortfall).

$$\mathrm{ES}_X(p) = \frac{1}{1-p} \int_p^1 \mathrm{VaR}_X(s)\, ds.$$

If X is atomless (i.e., its df F is continuous), the two are **equal**.

Example: exponential distribution

Let $X \sim \text{Exp}(\lambda)$. Then

$$\text{VaR}_X(p) = -\frac{\ln(1-p)}{\lambda}$$

and we have

$$\text{TVaR}_X(p) = \frac{1}{1-p}\int_p^1 -\frac{\ln(1-s)}{\lambda}\,ds = \frac{1}{\lambda} - \frac{\ln(1-p)}{\lambda} = \mathbb{E}\{X\} + \text{VaR}_X(p).$$

VaR vs TVaR

- Pick the worst $1 - p$ outcomes. Then TVaR/ES is the **mean** of their losses, while VaR is the **smallest loss** in that set.
- Hence, TVaR/ES takes into account the **severity** of losses in the tail, while VaR does not.

Distortion risk measures

Definition 5 (Distortion risk measure).
Given $X \sim F$ and a distribution function G such that[2] $0 \leq \alpha_G < \omega_G = 1$, the distortion risk measure is defined as

$$D_X(G) = \int_0^\infty G(\overline{F}(x))\, dx.$$

Theorem 6.
Distortion risk measures are coherent[3] if and only if G is concave.

[2]Recall that α_G and ω_G are the lower and upper endpoints of F
[3]The only thing requiring proof is that $D_X(G)$ is subadditive

Examples of distortion risk measures

- VaR is a distortion risk measure with $G(x) = \mathbb{1}\{1 - p \leq x \leq 1\}$. Since G is not concave, VaR is not coherent, as we know.
- TVaR is a distortion risk measure with $G(x) = \min\{1, x/(1-p)\}$. Since G is concave, TVaR is coherent!

Economic Capital

Definition 7 (Economic Capital).
The economic capital associated to a risk measure ρ is defined as

$$\mathrm{EC}_X = \rho(X) - \mathbb{E}\{X\},$$

where ρ is a risk measure.

For example, if $\rho = \mathrm{TVaR}$, then

$$\mathrm{EC}_X(p) = \mathrm{TVaR}_X(p) - \mathbb{E}\{X\}.$$

Interpretation: if $p = 0.9998$, then $\mathrm{EC}_X(p)$ represents the capital that a financial institution needs to hold to be able to cover 9998 losses out of 10000.

Multivariate risk measures

Multivariate extensions of VaR and TVaR exist, but they are significantly more complicated than in the univariate case. For example,

Definition 8 (Multivariate conditional tail expectation (MCTE)).
Given (X_1, X_2) with finite expectationd, define

$$\begin{aligned} \mathrm{MCTE}_1(p) &= \mathbb{E}\left\{X_1 \mid X_1 > \mathrm{VaR}_{X_1}(p), X_2 > \mathrm{VaR}_{X_2}(p)\right\}, \\ \mathrm{MCTE}_2(p) &= \mathbb{E}\left\{X_2 \mid X_1 > \mathrm{VaR}_{X_1}(p), X_2 > \mathrm{VaR}_{X_2}(p)\right\}. \end{aligned}$$

The drawback is that explicit formulas are rarely available, even for simple models.

Marginal expected shortfall (MES)

Definition 9 (Marginal expected shortfall).
Given (X_1, X_2) with finite expectations, define

$$\mathrm{MES}_1(p) = \mathbb{E}\left\{X_1 \mid X_2 > \mathrm{VaR}_{X_2}(p)\right\},$$

and similarly for $\mathrm{MES}_2(p)$.

Example: if (X_1, X_2) is bivariate normal with $N(0,1)$ marginals and $\rho \in (-1,1)$, then

$$\mathrm{MES}_1(p) = \frac{\rho\varphi(\Phi^{-1}(p))}{1-p},$$

where φ and Φ are the pdf and cdf of the standard normal distribution, respectively.

Questions/exercises

- Is mean-variance risk measure $\rho(X) = \mathbb{E}\{X\} + \lambda \operatorname{Var}\{X\}$ coherent for $\lambda > 0$?
- What kind of "bad news" can VaR systematically ignore?
- Distortion risk measures **reweight tail probabilities**. What does "concave distortion" mean behaviorally (attitude toward tail events)? What would a convex distortion represent?
- If two portfolios have the same VaR at level p but different ES at level p, what must be different about their loss distributions?