Lecture 14. Risk Aggregation[1]

(Optional)

Enkelejd Hashorva & Pavel Ievlev
Université de Lausanne

15 December, 2025

[1] This lecture is not part of the exam.

Learning objectives

- Understand why risk aggregation matters for capital requirements
- Compute aggregated risk for Gaussian and elliptical distributions
- Apply VaR and TVaR additivity for comonotonic risks
- Use comonotonic bounds when the dependence structure is unknown

Roadmap

1. **The aggregation problem** — why computing the distribution of $S = X_1 + \cdots + X_k$ is difficult
2. **Tractable case I: Gaussian & elliptical risks** — closed-form aggregation via variance
3. **Tractable case II: Comonotonic risks** — VaR and TVaR additivity
4. **Bounds for unknown dependence** — comonotonic upper bounds

Why do we aggregate risks?

Regulatory requirements: Solvency II, Basel III/IV require insurers and banks to hold capital against **total portfolio risk**, not individual risks.

Practical question: Given k business lines with risks $X_1, \ldots, X_k$, what is the capital requirement for the entire company?

$$\text{Required capital} = \rho(X_1 + \cdots + X_k) = \rho(S)$$

The challenge: We often know $\rho(X_i)$ for each line, but

$$\rho(S) \neq \rho(X_1) + \cdots + \rho(X_k) \quad \text{in general!}$$

Diversification benefit: Usually $\rho(S) < \sum_i \rho(X_i)$ — but by how much?

Remark: The inverse problem (**disaggregation**) — breaking down top-level results to the portfolio level — is also important in practice.

Part 1: The Aggregation Problem

Why is risk aggregation challenging?

If $X_1, \dots, X_k$ are **independent**, then $S = \sum_{i \le k} X_i$ is a convolution

$$F_S = F_{X_1} * \cdots * F_{X_k}.$$

But even then, closed-form expressions for the convolution are **rarely available**!

If the copula C of $(X_1, \dots, X_k)$ is **not** the independence copula C_I, then S is determined from both the marginal df's and the copula C.

Typically, C is unknown — so what can we do?

Three approaches:

1. Assume a specific copula $\rightarrow$ Monte Carlo simulation
2. Assume a tractable distribution family (Gaussian, elliptical)
3. Derive bounds that hold for **any** dependence structure

Simulation approach: Aggregation of log-normal risks

In actuarial practice, risk aggregation is performed using **Monte Carlo simulations**.

If $\boldsymbol{X} = (X_1, \ldots, X_k)^\top \sim N(\mathbf{0}, \Sigma)$, then

$$\boldsymbol{Y} \stackrel{d}{=} \exp(\boldsymbol{X}) \sim \mathrm{LN}(\mathbf{0}, \Sigma)$$

is a **log-normal** random vector.

Log-normal risks are important for both insurance and finance applications.

Using simulations, it is possible to calculate the df of

$$a_1 Y_1 + \cdots + a_k Y_k$$

for constants a_i, $i \leq k$.

Part 2: Gaussian & Elliptical Risks

Why Gaussian risks are tractable

Key property: Linear combinations of jointly Gaussian random variables are Gaussian.

If $\boldsymbol{X} = (X_1, \ldots, X_k) \sim N(\boldsymbol{\mu}, \Sigma)$, then for any constants $a_1, \ldots, a_k$:

$$S = a_1 X_1 + \cdots + a_k X_k \sim N\left(\sum_i a_i \mu_i, \sum_{i,j} a_i a_j \sigma_{ij}\right)$$

Consequence: Risk measures like VaR and TVaR can be computed in **closed form**!

This extends to **elliptically symmetric** distributions (e.g., Student-t).

Aggregation of Gaussian risks

Let $\boldsymbol{X} = (X_1, \ldots, X_k) \sim N(\mathbf{0}, \Sigma)$ with $\Sigma \in \mathbb{R}^{k \times k}$ a covariance matrix.

For constants a_i, $i \leq k$, we have

$$a_1 X_1 + \cdots + a_k X_k \stackrel{d}{=} bV$$

with $V \sim N(0, 1)$.

Since $bV \sim N(0, b^2)$, the constant b is found by

$$b^2 = \mathrm{Var}\{a_1 X_1 + \cdots + a_k X_k\} = \sum_{1 \leq i,j \leq k} a_i a_j \sigma_{ij}$$

Example $(k = 2)$: $b^2 = a_1^2 \sigma_{11} + a_2^2 \sigma_{22} + 2 a_1 a_2 \sigma_{12}$

Aggregation of randomly scaled risks

If $\boldsymbol{X} = (X_1, \ldots, X_k)^\top \sim N(\mathbf{0}, \Sigma)$ as above, then for the **common shock model**

$$\boldsymbol{Y} = W\boldsymbol{X}$$

with $W > 0$ independent of $\boldsymbol{X}$, we have for constants a_i, $i \leq k$:

$$\begin{aligned} a_1 Y_1 + \cdots + a_k Y_k &\stackrel{d}{=} W(a_1 X_1 + \cdots + a_k X_k) \\ &\stackrel{d}{=} WbV, \quad V \sim N(0,1) \end{aligned}$$

A particular instance is the **Student** (or t) **distribution**.

Radial representation of Gaussian vectors

Given a $k \times k$ real matrix A such that $AA^\top = \Sigma$, then

$$\boldsymbol{X} \stackrel{d}{=} A\boldsymbol{Z}$$

with $\boldsymbol{Z}$ having independent $N(0,1)$ components.

Important fact: the radius is independent of the angles, i.e.,

$$R = \sqrt{Z_1^2 + \cdots + Z_k^2} \quad \text{is independent of} \quad \boldsymbol{U} := \left(\frac{Z_1}{R}, \ldots, \frac{Z_k}{R} \right)$$

Aggregation of spherically symmetric risks

Recall that if $\boldsymbol{U} = (U_1, \dots, U_k)$ is uniformly distributed on the unit sphere

$$\mathbb{S}^{k-1} = \left\{ \boldsymbol{x} \in \mathbb{R}^k : \sum_{i=1}^{k} x_i^2 = 1 \right\}$$

then

$$\boldsymbol{U} \stackrel{d}{=} \left(\frac{Z_1}{R}, \dots, \frac{Z_k}{R} \right), \quad R = \sqrt{Z_1^2 + \cdots + Z_k^2}$$

and $Z_1, \dots, Z_k$ are iid $N(0,1)$ rv's independent of R.

If a_i, $i \le k$ are real constants, then

$$a_1 U_1 + \cdots + a_k U_k \stackrel{d}{=} U_1 \sqrt{\sum_{i=1}^{k} a_i^2}$$

Aggregation of elliptical random vectors

Example (Gaussian risks): If W^2 is chi-square distributed with k degrees of freedom, then $\boldsymbol{O} = W\boldsymbol{U}$ is Gaussian with independent components.

Let $\boldsymbol{Y}$ be given by

$$\boldsymbol{Y} = AW\boldsymbol{U} = A\boldsymbol{O}$$

which has mean vector zero if $\mathbb{E}\{W\}$ is finite.

Recall that $\boldsymbol{Y}$ is called an **elliptical RV** and $W > 0$ is independent of $\boldsymbol{U} = (U_1, \ldots, U_k)$.

Aggregation of elliptical RV's is as easy as for the Gaussian case!

Part 3: Comonotonic Risks

Why comonotonic risks are tractable

Comonotonic risks are driven by a **single** source of randomness:

$$X_i = F_i^{-1}(U), \quad U \sim \text{Unif}(0,1)$$

Interpretation: All risks move together — when one is high, all are high.

This represents the **"worst-case" dependence** for aggregation:

- No diversification benefit
- Maximum possible correlation

Key result: VaR and TVaR are **additive** for comonotonic risks!

Aggregation of comonotonic risks

Let $X_i = h_i(U)$, $i \leq k$ with $U \sim \text{Unif}(0,1)$ and $h_1, \ldots, h_k$ some measurable functions.

For constants a_i, $i \leq k$:

$$S = a_1 X_1 + \cdots + a_k X_k \stackrel{d}{=} a_1 h_1(U) + \cdots + a_k h_k(U)$$

A simple instance is $h_i(U) = c_i U$, $c_i \in \mathbb{R}$, so

$$S \stackrel{d}{=} (c_1 a_1 + \cdots + c_k a_k) U$$

VaR of monotone transforms

If h is **monotone non-decreasing** and continuous, we have

$$\mathrm{VaR}_{h(X)}(p) = h(\mathrm{VaR}_X(p)), \quad p \in (0,1)$$

This is true also if h is not continuous, but **only left-continuous** (recall that a quantile function is always left-continuous).

Application: For any $h_1, \ldots, h_k$ which are monotone non-decreasing and left-continuous:

$$\mathrm{VaR}_{\sum_{1 \le i \le k} h_i(U)}(p) = \sum_{1 \le i \le k} \mathrm{VaR}_{h_i(U)}(p), \quad p \in (0,1)$$

since $h(x) = \sum_{1 \le i \le k} h_i(x)$ is monotone non-decreasing and left-continuous.

If F_i's are df's, then $h_i = F_i^{-1}$ satisfy these assumptions.

VaR additivity for comonotonic risks

Denoting by G the df of $S = \sum_{i=1}^{k} F_i^{-1}(U)$, then

$$G^{-1}(q) = \sum_{i=1}^{k} F_i^{-1}(q), \quad q \in (0,1)$$

which simply means

$$\mathrm{VaR}_S(q) = \sum_{i=1}^{k} \mathrm{VaR}_{X_i}(q)$$

This is the **comonotonic additivity property** of VaR as a risk measure.

The same holds for TVaR:

$$\mathrm{TVaR}_S(q) = \sum_{i=1}^{k} \mathrm{TVaR}_{X_i}(q)$$

Numerical example: Independence vs comonotonicity

Let $X_1, X_2 \sim \text{Exp}(1)$ (exponential with mean 1). We want $\text{VaR}_{X_1+X_2}(0.95)$.

Case 1: Independent risks

- $X_1 + X_2 \sim \text{Gamma}(2, 1)$ (Gamma distribution)
- $\text{VaR}_{X_1+X_2}(0.95) \approx 5.32$

Case 2: Comonotonic risks

- $\text{VaR}_{X_i}(0.95) = -\ln(0.05) \approx 3.00$
- $\text{VaR}_{X_1+X_2}(0.95) = 2 \times 3.00 = 6.00$

Conclusion: Comonotonic VaR is $\approx 13\%$ higher — no diversification benefit!

Part 4: Bounds for Unknown Dependence

Relaxing the goal

So far: We computed the **exact** distribution of S under special assumptions (Gaussian, comonotonic).

Reality: Often we only know the marginals $F_1, \ldots, F_k$, but not the copula.

Relaxed goal: Instead of the exact distribution, find **bounds** that hold for **any** dependence structure.

Why useful? Upper bounds give **conservative** capital requirements — safe even in the worst case.

Aggregation and comonotonic risks

Let X_1, X_2 be non-negative risks with marginal df's F_1, F_2. Recall

$$X_i \stackrel{d}{=} F_i^{-1}(U), \quad i = 1, 2$$

with $U \sim \text{Unif}(0, 1)$ and F_i^{-1} the quantile function of X_i.

Note: This representation does not hold jointly unless (X_1, X_2) is a comonotonic random vector.

Consider two models for aggregation:

$$S = X_1 + X_2 \quad \text{and} \quad S^* = F_1^{-1}(U) + F_2^{-1}(U) =: Y_1 + Y_2$$

S^* is simpler to calculate or simulate since only U is random.

Comonotonic bounds

Key insight: Comonotonic risks are "maximally dependent" — they provide a **worst-case upper bound** for any aggregation.

Why? For any risks (X_1, X_2) with given marginals, we have

$$(X_1, X_2) \preceq_{\text{corr}} (F_1^{-1}(U), F_2^{-1}(U))$$

in the **correlation order** (i.e., comonotonic has maximal correlation).

Consequence: For any $d > 0$,

$$\mathbb{E}\{(S-d)_+\} \leq \mathbb{E}\{(S^* - d)_+\} = \mathbb{E}\{(Y_1 + Y_2 - d)_+\}$$

Comonotonic sums and stop-loss transform

Let q be such that $d = F_{S^*}^{-1}(q)$ with $F_{S^*}^{-1}$ the quantile function of S^*.

For comonotonic risks, the stop-loss transform is

$$\mathbb{E}\left\{(Y_1 + Y_2 - d)_+\right\} = \sum_{i=1}^{2} \mathbb{E}\left\{(Y_i - d_i)_+\right\}$$

where

$$d_i = F_i^{-1}(q), \quad i = 1, 2$$

So we have

$$\mathbb{E}\left\{(S - d)_+\right\} \leq \mathbb{E}\left\{(Y_1 - d_1)_+\right\} + \mathbb{E}\left\{(Y_2 - d_2)_+\right\}$$

Key takeaways

- **Aggregation is hard** because the distribution of $S = X_1 + \cdots + X_k$ depends on the (often unknown) copula
- **Gaussian/elliptical risks:** Linear combinations remain in the same family

$$S \sim N\left(\sum_i \mu_i, \sum_{i,j} a_i a_j \sigma_{ij}\right) \implies \text{closed-form VaR, TVaR}$$

- **Comonotonic risks:** VaR and TVaR are additive

$$\mathrm{VaR}_S(p) = \sum_i \mathrm{VaR}_{X_i}(p)$$

- **Unknown dependence:** Comonotonic sum provides worst-case upper bound

Questions/exercises

- Why does diversification reduce risk for independent risks but not for comonotonic risks?
- We showed that Gaussian and comonotonic risks are "tractable." What do these two cases have in common that makes aggregation easy?
- In what sense is the comonotonic bound "conservative"? When might it be too conservative to be useful?
- Regulators (Basel III) now require banks to use TVaR instead of VaR for market risk. Why might this be a better choice?