Lecture 3. Multivariate Gaussian & Elliptically symmetric risks

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Leargning objectives

- Understanding the multivariate (d-dimensional) dfs & pdfs;
- Focus on the properties of Gaussian random vectors;
- Understand the radial representation of Gaussian and ellipticocally symmetric random vectors;

Joint dfs of random vectors

Definition 1 (Distribution of a random vector).

Let $\boldsymbol{X} = (X_1, \ldots, X_d)$ be a d-dimensional random vector. Its **joint df** is the function $F : \mathbb{R}^d \to [0, 1]$ defined by

$$F(\boldsymbol{x}) = \mathbb{P}\left\{X_1 \leq x_1, \ldots, X_d \leq x_d\right\} = \mathbb{P}\left\{\boldsymbol{X} \leq \boldsymbol{x}\right\}.$$

We say that f is the **joint pdf** of F if F admits the following representation:

$$F(\boldsymbol{x}) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_d} f(y_1, \ldots, y_d)\, dy_1 \ldots dy_d.$$

Examples of d-dimensional dfs

- $F = \prod_{i=1}^{d} F_i$ the **product df** (independent components);
- $F = \min_{i=1,\dots,d} F_i$ the **upper df** (complete positive dependence);
- If $D \subset \mathbb{R}^d$ is a set of positive and finite volume $|D|$, then

$$f_D(\boldsymbol{x}) = \mathbb{1}\{\boldsymbol{x} \in D\}/|D|$$

  is a valid pdf (uniform distribution on D);
- Important special case: $D = \mathbb{S}^{d-1} = \{\boldsymbol{x} \in \mathbb{R}^d : x_1^2 + \cdots + x_d^2 = 1\}$ (the unit sphere in $\mathbb{R}^d$);

Lower "df" is not a df unless $d = 2$

Remark. The following natural analogue of the **lower df**

$$F(\boldsymbol{x}) = \left(\sum_{i=1}^{d} F_i(\boldsymbol{x}) - 1 \right)_+$$

is **not** a valid df for $d \geq 3$.

Think what could "countermonotonicity" mean in d dimensions?

Not every function is a joint df

As in the bivariate case, the following conditions are clearly **necessary** for F to be a joint df:

(C1) F is non-decreasing in each argument;

(C2) F is right-continuous;

(C3) $\lim_{x_i \to -\infty} F(\boldsymbol{x}) = 0$ for all $i \leq d$ and $\lim_{\boldsymbol{x} \to \infty} F(\boldsymbol{x}) = 1$;

However, there is **one more property** that is necessary:

$$\Delta_{1,h_1}(h_1) \dots \Delta_{d,h_d} F(\boldsymbol{x}) \geq 0 \quad \text{for all} \quad \boldsymbol{h} \geq \mathbf{0},$$

where Δ_i is the difference operator defined by

$$\Delta_{i,h_i} F(\boldsymbol{x}) = F(x_1, \dots, x_{i-1}, x_i + h_i, x_{i+1}, \dots, x_d) - F(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_d)$$

This property is called the **Δ-monotonicity**.

Meaning of Δ-monotonicity

The following formula explains the meaning of Δ-monotonicity:

$$\Delta_{1,h_1} \ldots \Delta_{d,h_d} F(\boldsymbol{x}) = \mathbb{P}\left\{\boldsymbol{x} < \boldsymbol{X} \leq \boldsymbol{x} + \boldsymbol{h}\right\} \geq 0.$$

Recall that in the bivariate case we had

$$\begin{aligned}
&\Delta_{1,h_1}\Delta_{2,h_2}F(x_1,x_2) \\
&\qquad = F(x_1+h_1,x_2+h_2) - F(x_1+h_1,x_2) - F(x_1,x_2+h_2) + F(x_1,x_2) \geq 0.
\end{aligned}$$

Multivariate copulas

> **Definition 2 (Copula of a random vector).**
> Let $\boldsymbol{X} = (X_1, \ldots, X_d)$ be a random vector with continuous marginals $F_1, \ldots, F_d$. The df C of the random vector $(F_1(X_1), \ldots, F_d(X_d))$ is called the **copula** of $\boldsymbol{X}$.

By Smirnov's theorem, marginals of C are $\mathrm{Unif}(0, 1)$, which motivates the following abstract definition of a copula:

> **Definition 3 (Copula).**
> A **copula** is a d-dimensional df C with $\mathrm{Unif}(0, 1)$ marginals.

The two definitions are equivalent in the sense that every copula in the abstract sense is the copula of some random vector.

Examples of multivariate copulas

As in the bivariate case, we can define the **product (independence) copula**

$$C_I(\boldsymbol{u}) = \prod_{i=1}^{d} u_i, \quad \boldsymbol{u} \in [0,1]^d,$$

and the **upper copula**

$$C_U(\boldsymbol{u}) = \min_{i=1,\ldots,d} u_i, \quad \boldsymbol{u} \in [0,1]^d.$$

However, as noted above, **there is no notion of lower copula** for $d \geq 3$, because there is no notion of lower df.

The function $\left(\sum_{i=1}^{d} u_i - d + 1\right)_+$ plays some important role in the theory of copulas, but it is **not a copula** itself unless $d = 2$.

Covariance matrix

Definition 4 (Covariance matrix).

If $\boldsymbol{X} = (X_1, \ldots, X_d)$ is a random vector with finite second moments, then its **covariance matrix** Σ is the $d \times d$ matrix with entries

$$\Sigma_{ij} = \operatorname{Cov}(X_i, X_j) = \mathbb{E}\left\{(X_i - \mathbb{E}\left\{X_i\right\})(X_j - \mathbb{E}\left\{X_j\right\})\right\}.$$

Gaussian random vectors

Definition 5 (Gaussian random vector).

- If $Z_1, \ldots, Z_d$ are **independent** $N(0,1)$ random variables, then $\boldsymbol{Z} = (Z_1, \ldots, Z_d)$ is called a **standard Gaussian random vector** in $\mathbb{R}^d$. This is denoted by $\boldsymbol{Z} \sim N_d(0, I)$, where I is the $d \times d$ identity matrix.
- If $\boldsymbol{\mu} \in \mathbb{R}^d$ is a vector and $A \in \mathbb{R}^{d \times d}$ is a matrix, then $\boldsymbol{X} = \boldsymbol{\mu} + A\boldsymbol{Z}$ is called a **Gaussian random vector** in $\mathbb{R}^d$ with **mean vector** $\boldsymbol{\mu}$ and **covariance matrix** $\Sigma = AA^\top$. This is denoted by $Y \sim N_d(\boldsymbol{\mu}, \Sigma)$.

Moment generating function of a Gausian vector

Definition 6 (Moment generating function).
If $\boldsymbol{X}$ is a rangom vector, then its **mgf**[1] is the function m defined by

$$m(\boldsymbol{t}) = \mathbb{E}\left\{e^{\boldsymbol{t}^\top \boldsymbol{X}}\right\} = \mathbb{E}\left\{\exp\left(\sum_{i=1}^{d} t_i X_i\right)\right\}, \quad \boldsymbol{t} \in \mathbb{R}^d.$$

Theorem 7.
If $Y \sim N(\boldsymbol{\mu}, \Sigma)$, *then* $m(\boldsymbol{t}) = \exp\left(\boldsymbol{t}^\top \boldsymbol{\mu} + \frac{1}{2}\boldsymbol{t}^\top \Sigma\, \boldsymbol{t}\right)$.

[1] **mgf** = moment generating function

Remarks on the mgf of a Gaussian vector

Remark 1. The mgf uniquely determines the distribution of $\boldsymbol{X}$ (if it exists in a neighborhood of $\mathbf{0}$).

Remark 2. Note that the formula depends on Σ, not on A. Hence, the laws of

$$\boldsymbol{Y} = \mu + A\boldsymbol{Z} \quad \text{and} \quad \boldsymbol{Y}' = \mu + A'\boldsymbol{Z}$$

coincide if $AA^\top = A'A'^\top$.

Pdf of a Gaussian vector

If Σ is **non-singular (invertible)**, then the pdf of $Y \sim N_d(\boldsymbol{\mu}, \Sigma)$ is given by

$$\varphi_d(\boldsymbol{x}) = \frac{1}{(2\pi)^{d/2}|\Sigma|^{1/2}} \exp\left(-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^\top \Sigma^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right), \quad \boldsymbol{y} \in \mathbb{R}^d,$$

where $|\Sigma| = \det(\Sigma) > 0$ is the determinant of Σ.

If Σ is **singular**, then $f_{\boldsymbol{Y}}$ does not exist (the distribution of Y is **concentrated on a hyperplane**[2] in $\mathbb{R}^d$).

[2] Imagine how it looks like in $d = 2$ and $d = 3$.

Bivariate Gaussian pdf

If $\boldsymbol{Y} \sim N(\mathbf{0}, \Sigma)$, the marginals are unit Gaussian ($Y_1, Y_2 \sim N(0,1)$), and the correlation between Y_1 and Y_2 is $\rho \in \boldsymbol{(-1,1)}$, then the matrix Σ is given by

$$\Sigma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$$

and Gaussian pdf simplifies to

$$\varphi_2(x_1, x_2) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{x_1^2 - 2\rho x_1 x_2 + x_2^2}{2(1-\rho^2)}\right).$$

If $\boldsymbol{\rho = \pm 1}$, then Σ is singular and the distribution of $\boldsymbol{Y}$ is **concentrated on the line**

$$\{(x,y) : y = \pm x\},$$

so **the pdf does not exist**.

Conditional distributions of a bivariate Gaussian vector

Let $\boldsymbol{Y} \sim N(\mathbf{0}, \Sigma)$ be a centered bivariate Gaussian vector. As mentioned above, there are many ways to factorize Σ as $AA^\top$. One possible choice is[3]

$$A = \begin{pmatrix} \sigma_1 & 0 \\ \rho\sigma_2 & \sigma_2\sqrt{1-\rho^2} \end{pmatrix}.$$

Therefore, with $\boldsymbol{Z} \sim N_2(\mathbf{0}, I)$ we have

$$\boldsymbol{X} = A\boldsymbol{Z} = \begin{pmatrix} \sigma_1 Z_1 \\ \rho\sigma_2 Z_1 + \sigma_2\sqrt{1-\rho^2} Z_2. \end{pmatrix}$$

Hence, the conditional distribution of X_2 given $X_1 = x_1$ is

$$(X_2 \mid X_1 = x_1) \sim N\left(\frac{\rho\sigma_2}{\sigma_1} x_1, \sigma_2^2(1-\rho^2)\right).$$

[3]Check that $AA^\top = \Sigma$.

Radial representation of a Gaussian vector

Let $\boldsymbol{Z}$ be a **standard Gaussian vector**: $\boldsymbol{Z} \sim N(\boldsymbol{0}, I)$. Define its **length** (or **radius**) by

$$R = \sqrt{Z_1^2 + \cdots + Z_d^2}.$$

Theorem 8 (Radial representation of a Gaussian vector).

- $R^2 \sim \text{Gamma}(\frac{d}{2}, \frac{1}{2})$.
- $\boldsymbol{U} = \boldsymbol{Z}/R \sim \text{Unif}(\mathbb{S}^{d-1})$.
- $\boldsymbol{U}$ *is independent of* R.
- *Hence,* $\boldsymbol{Z}$ *may be written as* $\boldsymbol{Z} = R\boldsymbol{U}$, *where* R *is an* $\mathbb{R}^1$ *random variable independent of* $\boldsymbol{U} \sim \text{Unif}(\mathbb{S}^{d-1})$.
- *If* $\boldsymbol{X} \sim N_d(\boldsymbol{\mu}, \Sigma)$, *then* $X \stackrel{d}{=} \boldsymbol{\mu} + RA\boldsymbol{U}$ *with any*[4] A *such that* $\Sigma = AA^\top$.

[4] A always exists, but is not unique

Elliptically symmetric random vectors

The radial representation of a Gaussian vector motivates the following definition:

Definition 9 (Elliptically symmetric random vector).
A random vector $\boldsymbol{X}$ in $\mathbb{R}^d$ is called **elliptically symmetric** if it admits the representation

$$\boldsymbol{X} \overset{d}{=} \boldsymbol{\mu} + RA\boldsymbol{U},$$

where $\boldsymbol{\mu} \in \mathbb{R}^d$, $A \in \mathbb{R}^{d\times d}$, $R \geq 0$ is an $\mathbb{R}^1$ random variable[5], and $U \sim \mathrm{Unif}(\mathbb{S}^{d-1})$ is independent of R. This is denoted by $\boldsymbol{X} \sim E_d(\boldsymbol{\mu}, A, H)$, where H is the df of R. The matrix $\Sigma = AA^\top$ is called the **dispersion matrix**[6] of $\boldsymbol{X}$.

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[5] Not necessarily Gamma$(\frac{d}{2}, \frac{1}{2})$

[6] This is not a covariance matrix unless $\boldsymbol{X}$ is Gaussian.

Examples of elliptical distributions

- Let $S > 0$ be a random variable independent of $V^2 \sim \text{Gamma}(\frac{d}{2}, \frac{1}{2})$. Define $R = SV$. This gives an elliptical random vector $\boldsymbol{X}$ called the **(scale) mixture of a Gaussian random vector**.
- If $R^2 = \alpha/Y$, where $Y \sim \text{Gamma}(\frac{\alpha}{2}, \frac{1}{2})$, then $\boldsymbol{Y}$ is called a **multivariate** t**-distribution (Student distribution)** with α degrees of freedom.

Exercises/questions

- Calculate $\Delta_{1,h_1}\Delta_{2,h_2}F$ for a bivariate df F by hand and check that it agrees with the formula from Lecture 1.
- If $(X_1, X_2, X_3) \sim F$ has pdf f, what is the df of (X_1, X_2) and does it have a pdf?
- Describe the distribution $N(\mu, 0)$. If $Y \sim N(a, \sigma^2)$, then is (μ, Y) a Gaussian vector?
- Consider $\boldsymbol{Y} = RA\boldsymbol{U}$, where $R = SV$, S and V are independent and $V^2 \sim \text{Gamma}(\frac{d}{2}, \frac{1}{2})$. Find the covariance matrix of $\boldsymbol{Y}$ in terms of $\mathbb{E}\left\{S^2\right\}$ and matrix A.