Lecture 4. Discrete mixtures of random vectors

Enkelejd Hashorva & Pavel Ievlev
Université de Lausanne

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Leargning objectives

- Introduce the class of discrete mixtures of random vectors
- Discuss the properties of discrete mixtures (pdfs, moments, etc)
- Understand mixture copulas and copulas of mixtures
- Discuss the dfs of random maxima, minima and sum

Mixtures of dfs

Definition 1.

- Consider a family of d-dimensional dfs $F_k : \mathbb{R}^d \to [0,1]$ parametrized by[1] $k = 1, \dots, n$, where $n \le \infty$.
- Let p_k be non-negative weights such that $\sum_{k=1}^n p_k = 1$. In other words, p_k is a pmf on $\{1, \dots, n\}$.
- The **mixture** of dfs F_k with weights p_k is a function $F : \mathbb{R}^d \to [0,1]$ defined by

$$F(\boldsymbol{x}) = \sum_{k=1}^{n} p_k F_k(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^d.$$

Exercise. Show that F is a valid df by checking the properties.

[1] k in the subsctipt does not denote k-th marginal here.

Examples of mixtures

Example. Laplace mixture:

$$F(\boldsymbol{x}) = \frac{1}{n}\sum_{k=1}^{n} F_k(\boldsymbol{x}).$$

Example. Convex sum of two dfs:

$$F(\boldsymbol{x}) = \theta F_1(\boldsymbol{x}) + (1-\theta)F_2(\boldsymbol{x}), \quad \theta \in [0,1].$$

Note that for $\theta \notin [0,1]$ the function F defined by this formula **is not a df**[2].

[2] Why?

Mixture of survival functions

Similarly to what we did for dfs, we can define the mixture of **survival functions** by

$$\bar{F}(\boldsymbol{x}) = \sum_{k=1}^{n} p_k \bar{F}_k(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^d.$$

Natural question. Is $\bar{F}$ the survival function of the mixture df F with the same weights p_k?

Pdfs of mixtures

Theorem 2 (Mixture pdf).

If for each k, F_k has a pdf f_k, then the mixture df F has a pdf f given by

$$f(\boldsymbol{x}) = \sum_{k=1}^{n} p_k f_k(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^d.$$

*If **at least one** F_k does not have a pdf, then neither does F (without proof).*

Proof. By definition of pdf, we have

$$F_k(\boldsymbol{x}) = \int_{\boldsymbol{y} \le \boldsymbol{x}} f_k(\boldsymbol{y})\, d\boldsymbol{y}.$$

Thus,

$$F(\boldsymbol{x}) = \sum_{k=1}^{n} p_k \int_{\boldsymbol{y} \le \boldsymbol{x}} f_k(\boldsymbol{y})\, d\boldsymbol{y} = \int_{\boldsymbol{y} \le \boldsymbol{x}} \sum_{k=1}^{n} p_k f_k(\boldsymbol{y})\, d\boldsymbol{y} = \int_{\boldsymbol{y} \le \boldsymbol{x}} f(\boldsymbol{y})\, d\boldsymbol{y}. \quad \square$$

Alternative proof. Differentiate[3]

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How to simulate from a mixture?

Algorithm. To simulate $\boldsymbol{X} \sim F$, where F is a mixture of F_k with weights p_k:

(i) Simulate K from the pmf p_k, $k = 1, \ldots, n$;

(ii) Simulate $\boldsymbol{X}$ from F_K.

Then, $\boldsymbol{X} \sim F$.

In other words, first choose k randomly according to p_k, then simulate from F_k.

We can write this as

$$\boldsymbol{X}_k \sim F_k \quad \text{and} \quad K \sim (p_k)_{k=1,\ldots,n} \implies \boldsymbol{X}_K \sim F.$$

Exercise. Use this as an alternative way to prove that mixture sf is the sf of the mixture df and vice versa.

Proof that the algorithm works

We only need to show that $\boldsymbol{X}_K$ is indeed distributed as F. To this end, we need to compute the df of $\boldsymbol{X}_K$. By the law of total probability, for any $\boldsymbol{x} \in \mathbb{R}^d$,

$$\begin{aligned}
\mathbb{P}\{\boldsymbol{X}_K \le \boldsymbol{x}\} &= \sum_{k=1}^{n} \mathbb{P}\{\boldsymbol{X}_K \le \boldsymbol{x} \mid K = k\}\,\mathbb{P}\{K = k\} \\
&= \sum_{k=1}^{n} \mathbb{P}\{\boldsymbol{X}_k \le \boldsymbol{x}\}\,p_k \\
&= \sum_{k=1}^{n} F_k(\boldsymbol{x})\,p_k = F(\boldsymbol{x}).
\end{aligned}$$

Mixture of copulas

Let C_k be the copula of F_k, $k = 1, \ldots, n$. The **mixture copula** C is defined by

$$C(\boldsymbol{u}) = \sum_{k=1}^{n} p_k C_k(\boldsymbol{u}), \quad \boldsymbol{u} \in [0,1]^d.$$

Question: What do we need to check to show that C is a valid copula?

Answer: Since each C_k is a df, C is a df as well. Hence, we only need to check that the marginals of C are uniform on $[0,1]$.

Exercise. Check that the marginals are indeed $\mathrm{Unif}(0,1)$.

Copula of mixture vs. mixture of copulas

Question. Is the copula of the mixture df F equal to the mixture copula C with the same weights p_k?

Answer. Not unless all F_k have the **same** marginals. This is a **very special case** and it's due to the fact that in his case the marginals[4] of F are the same as those of F_k. Check the exercise sessions for more details.

[4] Keep track of the indices!

Copula of mixture vs. mixture of copulas (contd.)

Remark. We can show that if F_k are dfs, then the **mixture of their copulas** is the df of

$$(F_{K,1}(X_{K,1}), \ldots, F_{K,d}(X_{K,d})),$$

where $K \sim (p_k)_{k=1,\ldots,n}$ and $\boldsymbol{X}_k \sim F_k$, whereas the **copula of the mixture df** is the df of

$$\mathbb{E}_{K'}\left\{(F_{K',1}(X_{K,1}), \ldots, F_{K',d}(X_{K,d}))\right\},$$

where K' is an independent copy of K.

Moments and mixtures

Theorem 3 (Moments of mixtures are mixtures of moments).
If $\boldsymbol{X} \sim F$*, where* F *is a mixture of* F_k *with weights* p_k*, then for all*[5] *functions* $g : \mathbb{R}^d \to \mathbb{R}$ *holds*

$$\mathbb{E}\left\{g(\boldsymbol{X})\right\} = \sum_{k=1}^{n} p_k \, \mathbb{E}\left\{g(\boldsymbol{X}_k)\right\}.$$

In particular,

$$\mathbb{E}\left\{X_1^{a_1} \cdots X_d^{a_d}\right\} = \sum_{k=1}^{n} p_k \, \mathbb{E}\left\{X_{k,1}^{a_1} \cdots X_{k,d}^{a_d}\right\}$$

for all $\boldsymbol{a}$ *such that the moments exist.*

[5] sufficiently nice

Proof of the previous theorem

$$\mathbb{E}\left\{g(\boldsymbol{X})\right\} = \int_{\mathbb{R}^d} g(\boldsymbol{x})\, dF(\boldsymbol{x}) = \int_{\mathbb{R}^d} g(\boldsymbol{x})\, d\left(\sum_{k=1}^{n} p_k F_k(\boldsymbol{x})\right)$$

$$= \sum_{k=1}^{n} p_k \int_{\mathbb{R}^d} g(\boldsymbol{x})\, dF_k(\boldsymbol{x}) = \sum_{k=1}^{n} p_k\, \mathbb{E}\left\{g(\boldsymbol{X}_k)\right\}.$$

Componentwise maxima

Let $\boldsymbol{X}_i$, $i = 1, \ldots, n$ be iid random vectors with joint df F. Define the **componentwise maxima** by

$$M_{i,n} = \max_{j=1,\ldots,n} X_{j,i}.$$

Theorem 4 (Df of the componentwise maxima).
The df of $\boldsymbol{M}_n = (M_{i,n})_{i=1,\ldots,n}$ *is* $F(\boldsymbol{x}) = F^n(\boldsymbol{x})$.

Proof. By definition of $\boldsymbol{M}_n$ and independence,

$$\begin{aligned} \mathbb{P}\left\{\boldsymbol{M}_n \leq \boldsymbol{x}\right\} &= \mathbb{P}\left\{X_{j,i} \leq x_i,\, j = 1, \ldots, n,\, i = 1, \ldots, d\right\} \\ &= \prod_{j=1}^{n} \mathbb{P}\left\{X_{j,i} \leq x_i,\, i = 1, \ldots, d\right\} = F^n(\boldsymbol{x}). \quad \square \end{aligned}$$

Random maxima is a mixture

Theorem 5 (Random maxima/minima is a mixture).
If K is a random variable with pmf $\mathbb{P}\{K = k\} = p_k$, $k = 1, \ldots, n$, then the ***df*** *of $\boldsymbol{M}_K = (M_{K,i})_{i=1,\ldots,n}$ is the mixture of dfs F^k with weights p_k.*

Similarly, the ***survival function*** *of the componentwise minima $\boldsymbol{m}_K = (m_{K,i})_{i=1,\ldots,n}$, where $m_{k,i} = \min_{j=1,\ldots,k} X_{j,i}$, is the mixture of* ***survival functions*** *$\bar{F}^k$ with weights p_k.*

Exercise. Prove this theorem using the law of total probability.

Random walk as a mixture

Theorem 6.
Let $\boldsymbol{X}_i \sim F$ be an iid sequence of random vectors and K be a random variable with pmf p_k. Then the df of the ***random walk stopped at a random time*** *K*

$$\boldsymbol{S}_K = \sum_{i=1}^{K} \boldsymbol{X}_i$$

is the mixture of dfs F^{*k} with weights p_k, where F^{*k} is the k-fold convolution of F with itself.

Exercise. Prove this theorem using the law of total probability and the fact that

$$F^{*k}(\boldsymbol{x}) = \mathbb{P}\left\{\boldsymbol{S}_k \leq \boldsymbol{x}\right\}.$$

Questions/exercises

- How can we simulate from $F = \theta H + (1 - \theta)G$ if we know how to simulate from H and G?
- If we want to model $F = \theta H + (1 - \theta)G$, the mixing random variable K is usually not observed directly. Can we do something about it?