Lecture 5. Copulas and marginals

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Leargning objectives

- Understand how a df can be decomposed into its marginals and a copula
- ...and how to construct a df using marginals and a copula
- Discuss Sklar's theorem, which ensures that such a decomposition is unique
- Study the invariance properties of copulas

Quantile function

Definition 1 (Quantile function).
The **qf**[1] of a df F is defined as

$$F^{-1}(u) = \inf\{x \in \mathbb{R} : F(x) \geq u\}, \quad u \in (0,1).$$

The most important property[2] of the quantile function is the following:

$$F^{-1}(q) \leq x \iff q \leq F(x).$$

Another important property is that the qf $F^{-1}(q)$ is **left-continuous** and **non-decreasing** (follows directly from right-continuity and monotonicity of F).

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[1] **qf** = quantile function
[2] or an equivalent definition

How to transform a rv into a uniform?

Recall that in the first lecture we stated the following result:

$$F(X) \sim \text{Unif}(0,1).$$

The proof was based on the following chain of equalities:

$$\mathbb{P}\{F(X) \leq u\} = \mathbb{P}\left\{X \leq F^{-1}(u)\right\} = F(F^{-1}(u)) = \neq u.$$

Question: where does this proof requires F to be **continuous**?

Answer: the equality $F(F^{-1}(u)) = u$ may fail if F is not continuous.

Important remark: $F^{-1}(U) \sim X$ if $U \sim \text{Unif}(0,1)$ holds **regardless** of whether F is continuous or not.

How to transform a rv into a uniform? (Contd.)

Since $F(X)$ is not necessarily $\mathrm{Unif}(0,1)$ if F is not continuous, is there still a way to transform X into a uniform rv?

Answer: yes, there is! Let $V \sim \mathrm{Unif}(0,1)$ be independent of $X \sim F$ and define

$$D_{V,F}(X) = F(X-) + \Big(F(X) - F(X-)\Big)V.$$

Theorem 2 (Transformation to uniform[3]).
The random variable $D_{V,F}(X)$ defined above is $\mathrm{Unif}(0,1)$.

[3] Without proof

Rosenblatt transform

Similarly to how the Smirnov transform **removes marginal information**, the **Rosenblatt transform** removes **dependence**. Let $(X, Y) \sim F$ and denote by $F_{2|1}(\cdot \mid x)$ the df of the **conditional** random variable $(Y \mid X = x)$.

Definition 3 (Rosenblatt transform).

Generate $V_1, V_2 \sim \mathrm{Unif}(0, 1)$ independently of (X, Y) and of each other. The **Rosenblatt transform** of (X, Y) is a vector (U_1, U_2) defined as

$$(U_1, U_2) = \big(D_{V_1,F_1}(X), D_{V_2,F_{2|1}(\cdot|X)}(Y)\big).$$

If F_1 and F_2 are continuous, the Rosenblatt transform simplifies to

$$(U_1, U_2) = \big(F_1(X), F_{2|1}(Y \mid X)\big).$$

Rosenblatt transform (Contd.)

The Rosenblatt transform has the following important property:

Theorem 4 (Rosenblatt transform to uniform[4]).
The random vector (U_1, U_2) *defined above is such that* $U_1, U_2 \sim \text{Unif}(0,1)$ *and are* ***independent****.*

[4] Without proof

How to simulate from a bivariate copula?

Let C be a bivariate copula and $(U_1, U_2) \sim C$. How to simulate from C?

Answer: use the Rosenblatt transform **in reverse**:

- Simulate $U_1, V_2 \sim \text{Unif}(0,1)$ independently.
- Set $U_2 = C_{2|1}^{-1}(V_2 \mid U_1)$.
- Return (U_1, U_2).

The conditional df $C_{2|1}(\cdot \mid u_1)$ is given by[5]

$$C_{2|1}(u_2 \mid u_1) = \mathbb{P}\left\{U_2 \leq v \mid U_1 = u_1\right\} = \frac{\partial C(u_1, u_2)}{\partial u_1}.$$

[5] See exercise sessions for the proof.

Copula + marginals → joint df

Let C be a copula and F_1, F_2 be two dfs. Define a function F by

$$\boxed{F(x_1, x_2) = C(F_1(x_1), F_2(x_2)).}$$

This is a **valid df** with marginals F_1, F_2 and copula C.

One way to see this is to take $(U_1, U_2) \sim C$ and set

$$X_1 = F_1^{-1}(U_1), \quad X_2 = F_2^{-1}(U_2)$$

and prove that $(X_1, X_2) \sim F$. Then, F is a df with marginals F_1 and F_2 *by construction.*

Sklar's theorem

Theorem 5 (Sklar's theorem).

Let F be a df with marginals F_i, $i = 1, \ldots, d$. Then, there exists a copula $C : [0,1]^d \to [0,1]$ such that

$$F(\boldsymbol{x}) = C(F_1(x_1), \ldots, F_d(x_d)).$$

*If all marginals F_i are continuous, then C is **unique** and is given by*

$$\boxed{C(\boldsymbol{u}) = F(F_1^{-1}(u_1), \ldots, F_d^{-1}(u_d)).}$$

Example: Gaussian copula

Let Φ and Φ^{-1} denote the df and qf of $N(0,1)$ and let Φ_2 denote the df of $N_2(\boldsymbol{\mu}, \Sigma)$ with any $\boldsymbol{\mu}$ and correlation $\rho \in (-1,1)$.

> **Definition 6 (Gaussian copula).**
> The **Gaussian copula** with parameter $\rho \in (-1,1)$ is defined as
>
> $$C_\rho(u_1, u_2) = \Phi_2(\Phi^{-1}(u_1), \Phi^{-1}(u_2)).$$

By plugging Φ_2 and Φ^{-1} into the above formula, we obtain

$$C_\rho(u_1, u_2) = \int_{-\infty}^{\Phi^{-1}(u_1)} \int_{-\infty}^{\Phi^{-1}(u_2)} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{s^2 - 2\rho st + t^2}{2(1-\rho^2)}\right) dt\, ds.$$

Remarks about the Gaussian copula

Although Φ_2 depends on $\boldsymbol{\mu}$ and the variances of Σ (so, $2+3=5$ parameters), the copula C_ρ only depends on the correlation ρ (1 parameter).

Similarly, the d-dimensional Gaussian copula

$$C(\boldsymbol{u}) = \Phi_d(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_d))$$

only depends on the $d(d-1)/2$ correlation parameters, while the d-variate normal distribution depends on d means and $d(d+1)/2$ variance and covariance parameters (a total of $d(d+3)/2$ parameters).

pdf of $F \to$ pdfs of C and F_i

Theorem 7 (Pdf of a df in terms of its copula and marginals).
If F has a pdf f, then so do the marginals F_i, as well as its copula C. The pdf of F can be expressed as

$$f(\boldsymbol{x}) = c(F_1(x_1), \dots, F_d(x_d)) \prod_{i=1}^{d} f_i(x_i),$$

Proof. By chain rule,

$$f(\boldsymbol{x}) = \frac{\partial^d F(\boldsymbol{x})}{\partial x_1 \cdots \partial x_d} = c(F_1(x_1), \dots, F_d(x_d)) \prod_{i=1}^{d} f_i(x_i). \quad \square$$

Pdfs of F_i and $C \to$ pdf of F

Conversely, if F_i have pdfs f_i and C has a pdf c, then the function F defined by

$$F(\boldsymbol{x}) = C(F_1(x_1), \ldots, F_d(x_d))$$

has a pdf f given by

$$f(\boldsymbol{x}) = c(F_1(x_1), \ldots, F_d(x_d)) \prod_{i=1}^{d} f_i(x_i).$$

Survival copula

Definition 8 (Survival copula).
Let C be a copula and $(U_1, U_2) \sim C$. Then the df of $(1 - U_1, 1 - U_2)$ denoted by Q is called the **survival copula** of C.

Important remark: as any df, C has a *survival function* $\overline{C}$. The survival **copula** Q is **not** that survival function: $Q \neq \overline{C}$!

The survival copula Q is still a copula, whereas $\overline{C}$ is not, because it's not even a df.

Transformation invariance

The following theorem is a **very powerful** result about copulas:

Theorem 9 (Transformation invariance[6]).

*Let $(X_1, X_2) \sim F$ with marginals F_1, F_2 and copula C, let $\boldsymbol{g} : \mathbb{R}^d \to \mathbb{R}$ be a **continuous** function, and let $(U_1, U_2) \sim C$.*

- *If both components g_i, $i = 1, 2$ are **increasing**, then the copula of $(g_1(X_1), g_2(X_2))$ is also C.*
- *If one component g_i is **increasing** and the other g_j, $j \neq i$ is **decreasing**, then the copula of $(g_1(X_1), g_2(X_2))$ is the copula of $(U_1, 1 - U_2)$.*
- *If both components g_i, $i = 1, 2$ are **decreasing**, then the copula of $(g_1(X_1), g_2(X_2))$ is the copula of $(1 - U_1, 1 - U_2)$, that is, the survival copula of C.*

*If the monotonicity of $\boldsymbol{g}$ is **strict**, the same results holds without continuity.*

[6] Without proof

Questions/exercises

- How can we simulate a Gaussian random vector?
- Find a formula for the survival copula Q of C in terms of C.

  Hint: rewrite Q as a survival *function* and use $\overline{F} = \overline{F}_1 + \overline{F}_2 + F - 1$.