Lecture 6. Max-stable distributions and copulas

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Leargning objectives

- Extend the notion of max-stability to multivariate dfs
- Discuss and characterize max-stable copulas

Univariate max-stable df

Definition 1 (Univariate max-stable df).

A univariate df F is called **max-stable** if for any integer $n \geq 1$ there exist constants $a_n > 0$ and $b_n \in \mathbb{R}$ such that

$$F^n(a_n x + b_n) = F(x), \quad x \in \mathbb{R}.$$

Recall that this is equivalent to saying that if $X_1, \ldots, X_n$ are iid random variables with df F, then

$$\max_{i=1,\ldots,n} X_i \stackrel{d}{=} a_n X_1 + b_n.$$

This motivates the name **max-stable**.

Motivation of max-stability

The class of max-stable dfs is important because of the following result.

Theorem 2.
If there exist sequences $a_n > 0$ and $b_n \in \mathbb{R}$ such that the df of the normalized maximum

$$\frac{\max_{i=1,\dots,n} X_i - b_n}{a_n} \xrightarrow{d} X$$

for some random variable X with non-degenerate df G, then G is max-stable.

- Thus, max-stable dfs characterize **all possible limit dfs of normalized maxima of iid random variables**.
- By the **Fisher-Tippett-Gnedenko theorem**, the only possible max-stable dfs are the **Gumbel** (Λ), **Fréchet** (Φ_α) and **Weibull** (Ψ_α) dfs.

Max-stable df

Definition 3 (Max-stable df).
A df F on $\mathbb{R}^d$ is called **max-stable** if for any integer $n \geq 1$ there exist vectors $\boldsymbol{a}_n > \mathbf{0}$ and $\boldsymbol{b}_n \in \mathbb{R}^d$ such that

$$F^n(\boldsymbol{a}_n \boldsymbol{x} + \boldsymbol{b}_n) = F(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^d.$$

Fixing x_j and letting $x_i \to \infty$ for all $i \neq j$ in the definition above, we obtain

$$F_j^n(a_{n,j} x_j + b_{n,j}) \leftarrow F^n(\boldsymbol{a}_n \boldsymbol{x} + \boldsymbol{b}_n) = F(\boldsymbol{x}) \to F_j(x_j).$$

Hence, **the marginals** of a max-stable df are **univariate max-stable dfs**.

Equivalent definition

Equivalently, F is max-stable if for any $n \geq 1$ we have

$$\max_{i=1,\dots,n} \boldsymbol{X}_i \stackrel{d}{=} \boldsymbol{a}_n \boldsymbol{X}_1 + \boldsymbol{b}_n,$$

where $\boldsymbol{X}_1, \dots, \boldsymbol{X}_n$ are iid random vectors with df F.

Max-stable marginals do not imply max-stable df

As we have just seen, F is max-stable $\implies$ its marginals are max-stable. What about the converse? If the marginals of F are max-stable, is F max-stable?

No, as the following example shows. Let F be a bivariate df with standard Fréchet marginals defined by

$$F(x,y) = \exp\left(-\frac{1}{x} - \frac{1}{y} - \frac{1}{xy}\right), \quad x, y > 0.$$

Then, for any $n \geq 1$,

$$F^n(nx, ny) = \exp\left(-\frac{1}{x} - \frac{1}{y} - \frac{n}{xy}\right) \neq F(x,y).$$

Question

Question: what additional conditions are needed for a df with max-stable marginals to be max-stable itself?

Answer: we need to impose some conditions on the **copula** of the df. We shall return to this question after discussing some examples.

Trivial examples

Some trivial examples of max-stable dfs are:

- The **independence df** $F(x_1, \ldots, x_d) = \prod_{i=1}^{d} F_i(x_i)$, where F_i are univariate max-stable dfs.
- The **comonotonicity df** $F(x_1, \ldots, x_d) = \min_{1 \leq i \leq d} F_i(x_i)$, where F_i are univariate max-stable dfs.

Exercise: Verify that the above dfs are indeed max-stable by checking the definition.

Exercise: Check that the **lower df** is not max-stable.

Example: Hüsler-Reiss df

Our first non-trivial example of a max-stable df is the bivariate **Hüsler-Reiss df**, which is defined as follows.

$$H_\lambda(x,y) = \exp\left(-e^{-x}\Phi\left(\lambda + \frac{y-x}{2\lambda}\right) - e^{-y}\Phi\left(\lambda + \frac{x-y}{2\lambda}\right)\right),$$

where $\lambda \in (0,\infty)$, $x, y \in \mathbb{R}$ and Φ the df of $N(0,1)$ law. The marginals of H_λ are standard Gumbel dfs:

$$H_\lambda(x,\infty) = \exp(-e^{-x}), \quad H_\lambda(\infty,y) = \exp(-e^{-y}).$$

With $a_n = 1$ and $b_n = \ln n$ we have

$$H_\lambda^n(x + \ln n, y + \ln n) = H_\lambda(x,y),$$

hence H_λ is max-stable.

Max-stable dfs with Fréchet marginals

If F has unit Fréchet marginals, i.e.,

$$F_i(x) = \Phi_1(x) = \exp(-1/x), \quad x > 0, \quad i = 1, \dots, d,$$

then max-stability is equivalent to the following simpler condition:

$$F^n(n\boldsymbol{x}) = F(\boldsymbol{x}) \quad \text{for all} \quad n \geq 1. F^t(t\boldsymbol{x}) = F(\boldsymbol{x}) \quad \text{for all} \quad t > 0.$$

De Haan representation

Theorem 4.
A df F on $\mathbb{R}^d$ with unit Fréchet marginals is max-stable if and only if there exists a random vector $\boldsymbol{Z} = (Z_1, \ldots, Z_d)$ with $Z_i \geq 0$ and $\mathbb{E}\{Z_i\} = 1$ for all $i = 1, \ldots, d$ such that F may be represented as

$$F(\boldsymbol{x}) = \exp\left(-\mathbb{E}\left\{\max_{1\leq i\leq d}\left(\frac{Z_i}{x_i}\right)\right\}\right), \quad \boldsymbol{x} > \boldsymbol{0}.$$

*The vector $\boldsymbol{Z}$ is called the **spectral random vector** of F.*

This theorem is useful for (a) constructing examples of max-stable dfs and (b) proving facts about them.

Elementary bounds from de Haan representation

Let $X, Y \geq 0$ be any random variables with $\mathbb{E}\{X\} = \mathbb{E}\{Y\} = 1$. Since the inequality $\max\{x, y\} \leq x + y$ holds for all x, y, we have

$$\mathbb{E}\left\{\max\left\{\frac{X}{x}, \frac{Y}{y}\right\}\right\} \leq \mathbb{E}\left\{\frac{X}{x} + \frac{Y}{y}\right\} = \frac{1}{x} + \frac{1}{y}.$$

Since $\mathbb{E}\{\max\{X, Y\}\} \geq \max\{\mathbb{E}\{X\}, \mathbb{E}\{Y\}\}$, we also have

$$\mathbb{E}\left\{\max\left\{\frac{X}{x}, \frac{Y}{y}\right\}\right\} \geq \max\left\{\frac{1}{x}, \frac{1}{y}\right\}.$$

Plugging this into the de Haan representation yields that any bivariate max-stable F with unit Fréchet marginals satisfies

$$\Phi_1(x)\Phi_1(y) \leq F(x, y) \leq \min\{\Phi_1(x), \Phi_1(y)\}$$

Copula of a max-stable df with unit Fréchet marginals

If F is a bivariate max-stable df with unit Fréchet marginals, then its copula C satisfies

$$C^n(u^{1/n}, v^{1/n}) = C(u, v).$$

Exercise: prove this using $C(u, v) = F(\Phi_1^{-1}(u), \Phi_2^{-1}(v))$ (proved in the last lecture), $F^n(nx, ny) = F(x, y)$ (max-stability) and explicit form of Φ_1^{-1}.

But the copula of C **does not depend on the marginals**! Therefore, the condition above holds for **any** max-stable copula, not only those corresponding to dfs with Fréchet marginals.

Max-stable copula

Definition 5 (Max-stable copula).
A copula C on $[0,1]^d$ is called **max-stable** if for any integer $n \geq 1$ holds

$$C^n(u_1^{1/n}, \ldots, u_d^{1/n}) = C(u_1, \ldots, u_d), \quad (u_1, \ldots, u_d) \in [0,1]^d.$$

Thus, a df F is max-stable, then its copula is max-stable and its marginals are univariate max-stable dfs.

The converse is also true: if the copula of F is max-stable and its marginals are univariate max-stable dfs, then F is max-stable.

Important remark

Even though the copula is a special case of a df, the notion of max-stability for copulas is **not** max-stability of df applied to a copula! Compare the definitions:

$$F^n(\boldsymbol{a}_n \boldsymbol{x} + \boldsymbol{b}_n) = F(\boldsymbol{x}) \quad \text{vs} \quad C^n(\boldsymbol{u}^{1/n}) = C(\boldsymbol{u}).$$

The definition of max-stable copula is designed to ensure that

$$F \text{ is max-stable} \iff \begin{cases} C \text{ is a max-stable copula} \\ \text{marginals of } F \text{ are univariate max-stable.} \end{cases}$$

Motivation of extreme value copulas

Similarly to how univariate max-stable dfs (def: stable under taking maximum) coincide with extreme valued dfs (def: possible limits of normalized maxima), multivariate max-stable dfs coincide with **extreme value max-stable dfs**.

And similarly to how univariate extreme value dfs were fully characterized by the Fisher-Tippett-Gnedenko theorem (three possible types), multivariate extreme value dfs are fully characterized by their **extreme value copulas** (i.e., max-stable copulas) and univariate extreme value marginals (i.e., univariate max-stable dfs).

$$F \text{ is extreme value df } \iff \begin{cases} C \text{ is an extreme value (max-stable) copula} \\ \text{marginals of } F \text{ are Fréchet, Gumbel or Weibull.} \end{cases}$$

It remains to give a **full characterization of extreme value copulas**, similar to three types of univariate max-stable dfs.

Bivariate extreme value copulas

Definition 6 (Bivariate extreme value copula).

A bivariate copula C is called an **extreme value copula** if it may be represented as

$$C_A(u,v) = (uv)^{A(\ln u/\ln(uv))},$$

where $A : [0,1] \to [1/2, 1]$ is a **convex** function satisfying

$$\max\{t, 1-t\} \leq A(t) \leq 1, \quad t \in [0,1].$$

The function A is called the **Pickands dependence function** of C.

In other words, the class of bivariate extreme value copulas is **parametrized** by Pickands dependence functions.

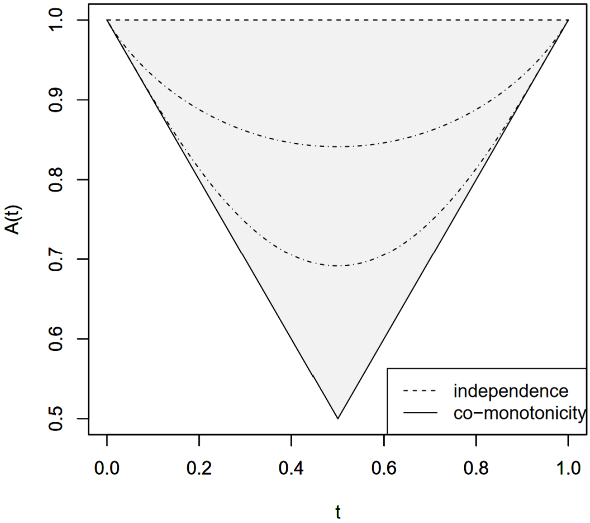
Max-stable = extreme value

Here's the promised characterization of bivariate max-stable copulas:

> **Theorem 7.**
> *A bivariate copula C is max-stable if and only if it is an extreme value copula, i.e., $C = C_A$ with some Pickands dependence function A.*

Therefore, a bivariate df F is max-stable if and only if its marginals are Fréchet, Gumbel or Weibull and its copula is C_A for some Pickands dependence function A.

How Pickands dependence functions look like?



Examples of Pickands dependence functions

We have already seen that the product (independence) df and the upper (comonotonicity) df are max-stable. Their copulas are[1] extreme value copulas with Pickands dependence functions

$$A^*(t) = 1, \quad A_*(t) = \max\{t, 1-t\}, \quad t \in [0,1],$$

respectively. Therefore[2],

$$\max\{t, 1-t\} \leq A(t) \leq 1A_*(t) \leq A(t) \leq A^*(t).$$

This implies all extreme value copulas C_A satisfy

$$C_I(u,v) \leq C_A(u,v) \leq C_U(u,v).$$

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[1] Check by plugging A into C_A.

[2] See picture in the previous slide.

Example: Marshall-Olkin copula

The **Marshall-Olkin copula** with parameters $\alpha, \beta \in (0,1)$ is defined as

$$C_{\alpha,\beta}(u,v) = \min\{u^{1-\alpha}v, uv^{1-\beta}\}, \quad (u,v) \in [0,1]^2.$$

It is easy to check that $C_{\alpha,\beta}$ is max-stable. Hence, it is an extreme value copula. We can show[3] that its Pickands dependence function is

$$A(t) = \max\{1-\alpha(1-t), 1-\beta t\}, \quad t \in [0,1].$$

[3] Check!

Example: Gumbel copula

The **Gumbel copula** with parameter $\theta \in [1, \infty)$ is defined as

$$C_\theta(u, v) = \exp\left(-\left((-\ln u)^\theta + (-\ln v)^\theta\right)^{1/\theta}\right), \quad (u, v) \in [0, 1]^2.$$

It is easy to check that C_θ is max-stable. Hence, it is an extreme value copula. We can show[4] that its Pickands dependence function is

$$A(t) = \left(t^\theta + (1 - t)^\theta\right)^{1/\theta}, \quad t \in [0, 1].$$

[4] Check!

Questions/exercises

- Where does $\max\{1-t,t\} \le A(t) \le 1$ condition come from? *Hint: we have proved some inequality for max-stable bivariate dfs with unit Fréchet marginals earlier today. Find this inequality and see what it says about the corresponding copula. Note that properties of copulas are independent of marginals to drop the unit Fréchet assumption.*
- If F is a df, is F^t for $t > 0$ a df? And if F is max-stable?
- What properties we can use to disprove that a given copula is max-stable/extreme value?
- If the mixtrure copula $C = \theta C_I + (1-\theta) C_L$ an extreme value copula?
- Is the Farlie-Gumbel-Morgenstern copula $C(u,v) = uv(1 + a(1-u)(1-v))$ an extreme value copula?