Lecture 7. Max-domains of attraction

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Leargning objectives

- Define max-domains of attraction
- Study the relation between MDAs, marginals and copulas
- Introduce the asymptotic independence property

Max-DOA of a distribution

Definition 1 (Max-domain of attraction).

A distribution function F is said to belong to the max-domain of attraction (MDA) of distribution function G, denoted by $F \in \mathrm{MDA}(G)$, if there exist sequences of vectors $\boldsymbol{a}_n > \mathbf{0}$ and $\boldsymbol{b}_n \in \mathbb{R}^d$ such that

$$F^n(\boldsymbol{a}_n \boldsymbol{x} + \boldsymbol{b}_n) \xrightarrow[n\to\infty]{} G(\boldsymbol{x})$$

for all continuity points $\boldsymbol{x}$ of G. Equivalently, if $\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots$ are i.i.d. random vectors with distribution function F, then

$$\frac{\max_{1\le i\le n} \boldsymbol{X}_i - \boldsymbol{b}_n}{\boldsymbol{a}_n} \xrightarrow[n\to\infty]{d} \boldsymbol{Z} \sim G.$$

Max-DOA and extreme value distributions

Theorem 2.
If $F \in MDA(G)$ for some non-degenerate distribution function G, then G is an extreme value distribution.

Proof sketch. First, we prove that that

$$\frac{\boldsymbol{a}_{mn}}{\boldsymbol{a}_n} \to \boldsymbol{\alpha}_m > \mathbf{0} \quad \text{and} \quad \frac{\boldsymbol{b}_{mn} - \boldsymbol{b}_n}{\boldsymbol{a}_n} \to \boldsymbol{\beta}_m \quad \text{as} \quad n \to \infty.$$

Then, we write

$$\begin{aligned} G^m(\boldsymbol{\alpha}_m \boldsymbol{x} + \boldsymbol{\beta}_m) &= \lim_{n\to\infty} F^{mn}(\boldsymbol{a}_n(\boldsymbol{\alpha}_m \boldsymbol{x} + \boldsymbol{\beta}_m) + \boldsymbol{b}_n) \\ &= \lim_{n\to\infty} F^{mn}(\boldsymbol{a}_{mn}\boldsymbol{x} + \boldsymbol{b}_{mn} + o(1)) = G(\boldsymbol{x}). \end{aligned}$$

Copula's MDA

Definition 3.
A copula C is said to belong to the max-domain of attraction of a copula Q, denoted by $C \in \mathrm{MDA}(Q)$, if

$$C^n(\boldsymbol{u}^{1/n}) \xrightarrow[n\to\infty]{} Q(\boldsymbol{u})$$

for all continuity points $\boldsymbol{u}$ of Q.

Theorem 4.
If $C \in MDA(Q)$ for some copula Q, then Q is an extreme value copula.

MDA in terms of marginals and copula

Theorem 5.
$F \in \mathrm{MDA}(G) \iff F_i \in \mathrm{MDA}(G_i)$ *for all* i *and* $C_F \in \mathrm{MDA}(C_G)$.

Proof sketch. First, assume that $F \in \mathrm{MDA}(G)$. Then, as we've already seen, $F_i \in \mathrm{MDA}(G_i)$ for all i. Next, we have

$$F_i^n(a_{n,i}x + b_{n,i}) \approx G_i(x) \implies F_i(x) \approx G_i^{1/n}\left(\frac{x - b_{n,i}}{a_{n,i}}\right)$$

$$\implies F_i^{-1}(u) \approx a_{n,i}G_i^{-1}(u^n) + b_{n,i}$$

$$\implies F_i^{-1}(u^{1/n}) \approx a_{n,i}G_i^{-1}(u) + b_{n,i}.$$

Therefore, denoting $\boldsymbol{F}^{-1}(\boldsymbol{x}) = (F_1^{-1}(x_1), \ldots, F_d^{-1}(x_d))$ we obtain

$$C_F^n(\boldsymbol{u}^{1/n}) = F^n(\boldsymbol{F}^{-1}(\boldsymbol{u}^{1/n})) \approx F^n(\boldsymbol{a}_n\boldsymbol{G}^{-1}(\boldsymbol{u}) + \boldsymbol{b}_n) \approx G(\boldsymbol{G}^{-1}(\boldsymbol{u})) = C_G(\boldsymbol{u}).$$

Tail dependence coefficient in terms of copula

Definition 6 (Tail dependence coefficient).
The upper tail dependence coefficient of a bivariate copula C is defined as

$$\lambda(C) = \lim_{u \uparrow 1} \frac{\mathbb{P}\{U > u,\ V > u\}}{1 - u},$$

where $(U, V) \sim C$, provided that the limit exists. The copula C is said to be asymptotically independent if $\lambda(C) = 0$.

Theorem 7.
C is asymptotically independent if and only if $C \in \mathrm{MDA}(C_I)$.

Thus, to check whether $C \in \mathrm{MDA}(C_I)$ we only need to compute one number $\lambda(C)$.

Independence implies asymptotic independence

Let us check that the interpretation of $\lambda(C) = 0$ as some kind of "independence" is consistent with the independence itself. That is, let us check that $\lambda(C_I) = 0$:

$$\lambda(C_I) = \lim_{u\uparrow 1} \frac{\mathbb{P}\{U > u,\ V > u\}}{1-u} = \lim_{u\uparrow 1} \frac{(1-u)^2}{1-u} = 0.$$

Moreover, if C is any copula such that $\bar{C} \leq \bar{C}_I$, then

$$0 \leq \lambda(C) = \lim_{u\uparrow 1} \frac{\mathbb{P}\{U > u,\ V > u\}}{1-u} = \lim_{u\uparrow 1} \frac{\overline{C}(u,u)}{1-u} \leq \lim_{u\uparrow 1} \frac{\overline{C_I}(u,u)}{1-u} = 0,$$

hence $\lambda(C) = 0$. For example, the **lower copula** C_L satisfies[1] $\lambda(C_L) = 0$.

[1] Check this directly.

Tail dependence coefficient in terms of df

If F is a bivariate df with equal[2] marginals $F_1 = F_2 = H$, then the tail dependence coefficient of C_F can be expressed in terms of F

$$\lambda(C_F) = \lim_{x \uparrow \omega} \frac{\overline{F}(x,x)}{\overline{H}(x)} = \lim_{x \uparrow \omega} \frac{\mathbb{P}\left\{X_1 > x, X_2 > x\right\}}{\mathbb{P}\left\{X_1 > x\right\}},$$

where $\omega \leq \infty$ is the upper endpoint of H. Indeed,

$$\lim_{x \uparrow \omega} \frac{\overline{F}(x,x)}{\overline{H}(x)} = \lim_{u = F(x) \uparrow 1} \frac{\overline{F}(H^{-1}(u), H^{-1}(u))}{\overline{H}(H^{-1}(u))} = \lim_{u \uparrow 1} \frac{\overline{C_F}(u,u)}{1-u} = \lambda(C_F).$$

[2] What's different when the marginals are not the same?

Example: Gaussian df is asymptotically independent

Let $X, Y \sim N(0,1)$ be jointly Gaussian with $|\rho| < 1$. Then, the tail dependence coefficient of the corresponding copula C_ρ is

$$
\begin{aligned}
\lambda(C_\rho) &= \lim_{x\to\infty} \frac{\mathbb{P}\{X > x, Y > x\}}{\mathbb{P}\{X > x\}} && \text{formula for } \lambda(C_F) \\
&\le \lim_{x\to\infty} \frac{\mathbb{P}\{X + Y > 2x\}}{\mathbb{P}\{X > x\}} && \left.\begin{smallmatrix} X>x \\ Y>x \end{smallmatrix}\right\} \implies X + Y > 2x \\
&= \lim_{x\to\infty} \frac{\varphi(2x/\sqrt{2(1+\rho)})}{\sqrt{2(1+\rho)}\,\varphi(x)} && \text{l'Hopital's rule and } \frac{d\overline{\Phi}(x)}{dx} = -\varphi(x) \\
&= \lim_{x\to\infty} \frac{\exp\left(-\frac{x^2}{1+\rho} + \frac{x^2}{2}\right)}{\sqrt{2(1+\rho)}} = 0 && \text{because } -\frac{1}{1+\rho} + \frac{1}{2} < 0
\end{aligned}
$$

For $\rho = -1$, $C_\rho = C_L$ and we already know that $\lambda(C_L) = 0$.

Example: elliptical distributions with Gumbel MDA marginals

Generalizing the previous example, one can show that if (X, Y) has a bivariate elliptical distribution with Gumbel MDA marginals (as Gaussian), then X and Y are asymptotically independent.

Tail dependence of EVC

Theorem 8.
If C_A is an extreme value copula with Pickands dependence function A, then its tail dependence coefficient is

$$\lambda(C_A) = 2(1 - A(1/2)).$$

Proof. We have

$$\begin{aligned}\overline{C}_A(u,u) &= 1 - 2u + C_A(u,u) && \text{formula for } \overline{C} \\ &= 1 - 2u + u^{2A(1/2)} && \text{by } C_A(u,v) = (uv)^{A(\ln u/\ln(uv))}\end{aligned}$$

By l'Hopital's rule,

$$\lambda(C_A) = \lim_{u\uparrow 1} \frac{\overline{C}_A(u,u)}{1-u} = \lim_{u\uparrow 1} \frac{-2 + 2A(1/2)u^{A(1/2)-1}}{-1} = 2(1 - A(1/2)).$$

Example: tail dependence of Gumbel copula

Calculating the tail dependence coefficient of the Gumbel copula

$$C_\theta(u,v) = \exp\left(-\left((-\ln u)^\theta + (-\ln v)^\theta\right)^{1/\theta}\right),$$

directly from the definition is challenging. However, since the Gumbel copula is an EVC with Pickands dependence function

$$A(t) = \left(t^\theta + (1-t)^\theta\right)^{1/\theta},$$

we can use the previous theorem to obtain

$$\lambda(C_\theta) = 2\left(1 - A(1/2)\right) = 2 - 2^{1/\theta}.$$

Questions/exercises

- Let $(X, Y) \sim \text{Unif}(\mathbb{S}^1)$. Is (X, Y) asymptotically independent?
- Let $(X, Y) \sim \text{Unif}(\{x^2 + y^2 < 1\})$. Is (X, Y) asymptotically independent?
- Why did we need to assume that the marginals are equal when expressing the tail dependence coefficient in terms of the df? What goes wrong if they aren't?