Lecture 8. Archimedean copulas

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Leargning objectives

- Introduce a special class of copulas called Archimedean copulas
- Discuss their properties and popular examples
- Explain the link between Archimedean copulas and frailty models

Archimedean generator and Archimedean copula

Definition 1 (Archimedean generator).

A function $\psi : (0, 1] \to [0, \infty)$ is called an **Archimedean generator** if

- $\psi(1) = 0$
- $\psi'(s) < 0$ for all $s \in (0, 1)$
- $\psi''(s) > 0$ for all $s \in (0, 1)$

Definition 2 (Archimedean copula).

Let ψ be an Archimedean generator. The **Archimedean copula** is defined by

$$C_\psi(\boldsymbol{u}) = \begin{cases} \psi^{-1}\left(\psi(u) + \psi(v)\right), & \text{if } \psi(u) + \psi(v) \leq \psi(0), \\ 0, & \text{otherwise.} \end{cases}$$

Remarks about Archimedean copula & generator

- The assumptions $\psi' < 0$ and $\psi'' > 0$ may be relaxed to ψ being strictly decreasing and convex.
- Note that ψ is not defined at 0, but we can put $\psi(0) = \lim_{s \downarrow 0} \psi(s)$. This limit exists by monotonicity, but may be equal to ∞.
- Since ψ is strictly decreasing, its inverse ψ^{-1} is well-defined on $[0, \psi(0))$, which is where the conditional in the definition of C_ψ comes from. We can also extend ψ^{-1} by $\psi^{-1}(\psi(0)) = 0$.
- We can write the same formula for C_ψ in a more compact way as follows:

$$C_\psi(\boldsymbol{u}) = \psi^{-1}\left(\max\left\{\psi(0), \psi(u) + \psi(v)\right\}\right).$$

Multivariate Archimedean copula

The definition of Archimedean copula can be extended to higher dimensions $d > 2$ as follows:

$$C_\psi(u_1, \ldots, u_d) = \psi^{-1}\left(\max\left\{\psi(0), \sum_{i=1}^{d} \psi(u_i)\right\}\right).$$

However, assuming that $\psi' < 0$ and $\psi'' > 0$ is **not sufficient** to guarantee that C_ψ is a copula for $d > 2$. The issue is with the Δ-monotonicity condition.

The sufficient[1] condition for C_ψ to be a d-dimensional is the following:

$$(-1)^k \psi^{(k)}(s) \geq 0, \quad \text{for all } s \in (0,1) \text{ and } k = 1, \ldots, d.$$

[1] and almost necessary

Example: independence copula is Archimedean

Let $\psi(x) = -\ln x$. Then ψ is an Archimedean generator since $\psi'(x) = -1/x < 0$ and $\psi''(x) = 1/x^2 > 0$. The corresponding bivariate Archimedean copula is

$$C_\psi(u, v) = \psi^{-1}(\psi(u) + \psi(v)) = \exp(-(-\ln u - \ln v)) = uv,$$

which is the independence copula.

Is Archimedean generator unique?

Question: is it possible that $C_\varphi = C_\psi$ with $\varphi \neq \psi$?

Answer: yes, for example, let $\varphi(x) = -c \ln x$ with $c > 0$. Then, $\varphi^{-1}(x) = e^{-x/c}$ and we have

$$C_\varphi(u, v) = \varphi^{-1}(\varphi(u) + \varphi(v)) = \exp\left(-\frac{-c\ln u - c\ln v}{c}\right) = uv.$$

However:

Theorem 3.
If $C_\psi = C_\varphi$, then there exists a constant $c > 0$ such that $\varphi = c\psi$.

Examples: Clayton, Gumbel & Frank

- The **Clayton** copula is Archimedean with generator $\psi_\theta(s) = (s^{-\theta} - 1)/\theta$ for $\theta > 0$.
- The **Gumbel** copula is Archimedean with generator $\psi_\theta(s) = (-\ln s)^\theta$ for $\theta \geq 1$.
- The **Frank** copula is Archimedean with generator $\psi_\theta(s) = -\ln\left(\frac{e^{-\theta s}-1}{e^{-\theta}-1}\right)$ for $\theta \neq 0$.

Basic properties of Archimedean copulas

- Archimedean copulas are **exchangeable**/symmetric/invariant under permutations of their arguments: $C_\psi(u, v) = C_\psi(v, u)$.
- Archimedean copulas are associative, i.e., for any u, v, w we have

$$C_\psi(C_\psi(u, v), w) = C_\psi(u, C_\psi(v, w)).$$

Tail dependence of Archimedean copulas

Theorem 4.
The upper tail dependence coefficient of an Archimedean copula C_ψ is given by

$$\lambda(C_\psi) = 2 - \lim_{x \downarrow 0} \frac{1 - \psi^{-1}(2x)}{1 - \psi^{-1}(x)}.$$

Proof. By change of variable $x = \psi(u) \downarrow 0$ if $u \uparrow 1$, we have

$$\lambda(C_\psi) = \lim_{u \uparrow 1} \frac{1 - 2u + C_\psi(u,u)}{1-u} = \lim_{u \uparrow 1} \frac{2(1-u) - 1 + \psi^{-1}(2\psi(u))}{1-u}.$$

$$= 2 - \lim_{x \downarrow 0} \frac{1 - \psi^{-1}(2x)}{1 - \psi^{-1}(x)}.$$

Characterization of Archimedean copulas

Take the following objects:

- $\boldsymbol{E} = (E_1, \dots, E_d)$ a random vector with iid **standard exponential** components
- $R > 0$ another random variable independent of E_i's

Define a random vector by $\boldsymbol{X} = R\boldsymbol{E}$.

Theorem 5.
A copula C is Archimedean if and only if it is the survival copula of a random vector $\boldsymbol{X}$ defined above for some random variable $R > 0$.

Archimedean copulas associated with Laplace transforms

Theorem 6.
Let $\Theta \geq 0$ be a random variable and $L(t) = \mathbb{E}\{e^{-t\Theta}\}$ its Laplace transform. Take two iid standard exponential random variables E_1, E_2 independent of Θ. Define $U = L(E_1/\Theta)$ and $V = L(E_2/\Theta)$. Then the df of (U, V) is a copula. Moreover, this copula is Archimedean with generator $\psi = L^{-1}$.

In other words, $(U, V) \sim C_\psi$ may be represented as

$$U = \psi^{-1}(E_1/\Theta), \quad V = \psi^{-1}(E_2/\Theta), \quad \text{where } \Theta \text{ is an rv with} \quad \mathbb{E}\{e^{-s\Theta}\} = \psi^{-1}(s).$$

Conditionally on Θ, U and V are independent.

Frailty model

- Let (X_1, X_2) be conditionally independent given "frailty variable[2]" Θ.
- Moreover, assume that the conditional marginals are

$$\mathbb{P}\{X_i \le x \mid \Theta = \theta\} = H_i^\theta(x)$$

  with some dfs $H_i, i = 1, 2$.
- Then, the joint df and marginals of (X_1, X_2) are given by

$$\mathbb{P}\{X_1 \le x_1,\ X_2 \le x_2\} = \mathbb{E}\{H_1^\Theta(x_1)\, H_2^\Theta(x_2)\}, \quad \mathbb{P}\{X_i \le x\} = \mathbb{E}\{H_i^\Theta(x)\}.$$

- Denote $\psi^{-1}(t) = \mathbb{E}\{e^{-t\Theta}\}$. Then $F_i(x_i) = \psi^{-1}(-\ln H_i(x_i))$.
- Moreover, $\mathbb{P}\{X_1 \le x_1,\ X_2 \le x_2\} = \psi^{-1}(-\ln H_1(x_1) - \ln H_2(x_2))$.
- Removing the marginals, we see that the copula of (X_1, X_2) is Archimedean with generator ψ.

[2] Interpretation: conditionally on (unobserved) frailty factor, the lifetimes are independent.

Questions/exercises

- Prove Theorem **??** by showing that $h = \psi \circ \varphi^{-1}$ satisfies the so-called Cauchy equation $h(x + y) = h(x) + h(y)$. The only continuous solution of this equation is $h(x) = cx$ with some $c \in \mathbb{R}$. The positivity of c follows from the monotonicity of ψ and φ.
- Calculate the Frank copula from its generator.
- Calculate $C(u, v) = F(F_i^{-1}(u), F_i^{-1}(v))$ in the frailty model to prove that it is Archimedean with generator ψ.
- If $\Theta \sim \text{Gamma}(\frac{1}{\alpha}, 1)$, then $L(t) = (1 + t)^{-1/\alpha}$. What is the corresponding Archimedean copula?