Theory sheet 2

Definition 1. *A permutation* π *of the set* $\{1,\ldots,n\}$ *is any* <u>*one-to-one*</u> *function*

$$\pi : \{1,\ldots,n\} \to \{1,\ldots,n\}.$$

- To define a permutation is to define its values on every element of $\{1,\ldots,n\}$. For example, $\pi(1) = 2$ and $\pi(2) = 1$ defines a permutation of $\{1,2\}$.
- The identity permutation is defined by $\pi(i) = i$ for all i.
- We say that there is an <u>inversion</u> of π if $i < j$, but $\pi(i) > \pi(j)$ (so π flips the ordering of i and j).
- A permutation is said to be <u>even/odd</u> if its number of inversions is even/odd.
- The <u>sign</u> of a permutation is defined by $\operatorname{sign}(\pi) = +1$ if π is even and $\operatorname{sign}(\pi) = -1$ if π is odd.

Definition 2. *Determinant of a* <u>*square*</u> *matrix* $A \in M_{n,n}(\mathbb{R})$ *is a the following number:*

$$\det A = \sum_{\pi:\ \textit{permutation of } \{1,\ldots,n\}} \operatorname{sign}(\pi) \cdot a_{1\pi(1)}\, a_{2\pi(2)} \cdots a_{n\pi(n)}.$$

Alternative notation for the determinant:

$$\det \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

- If $n = 1$, then $A = (a_{11})$ and $\det A = a_{11}$.
- If $n = 2$, then
$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = +a_{11}a_{22} - a_{12}a_{21}$$
- If $n = 3$, then
$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = +a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{32} + a_{13}a_{21}a_{32}$$
- There are $n! = 1 \cdot 2 \cdot \ldots \cdot n$ permutations of n elements. If $n = 3$, the number is 6, if $n = 4$, the number is 24. Hence, finding determinant by definition is very <u>inefficient</u>. There are better approaches.

- Here are two mnemonics for memorizing the formulas for $n = 2$ and $n = 3$:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = +a_{11}a_{22} - a_{12}a_{21}$$

and

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = +a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$

$$-a_{13}a_{21}a_{32} - a_{11}a_{32}a_{23} - a_{31}a_{22}a_{13}.$$

This is known as the Sarrus rule.

- For larger matrices, the best approach is using elementary transformations! See below.
- If the matrix contains a zero row/column, then $\det A = 0$. For example,

$$\det \begin{pmatrix} 3 & 4 & 5 \\ 0 & 0 & 0 \\ 7 & 1 & 9 \end{pmatrix} = 0.$$

- If A is diagonal, then $\det A =$ product of diagonal entries. For example,

$$\det \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 11 \end{pmatrix} = 3 \cdot 5 \cdot 7 \cdot 11 = 1155.$$

Moreover, if A is upper triangular or lower triangular, the same is true:

$$\det \begin{pmatrix} 3 & 1 & 4 & 8 \\ 0 & 5 & -2 & 16 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 11 \end{pmatrix} = 3 \cdot 5 \cdot 7 \cdot 11 = 1155.$$

- We have already seen that elementary transformations do not change rank. They do change the determinant, but in a controlled way:
  - Swapping two lines changes sign of the determinant:

$$\det \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \end{pmatrix} = -\det \begin{pmatrix} a_{21} & a_{22} & \cdots & a_{2n} \\ a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \end{pmatrix}$$

.

  - Multiplying one row by $\lambda \neq 0$ multiplies the determinant by λ:

$$\det \begin{pmatrix} \lambda a_{11} & \lambda a_{12} & \cdots & \lambda a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \end{pmatrix} = \lambda \det \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \end{pmatrix}.$$

- Adding a multiple of one line to the other does not change $\det A$:

$$\det\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \end{pmatrix} = \det\begin{pmatrix} a_{11}+\lambda a_{21} & a_{12}+\lambda a_{22} & \cdots & a_{1n}+\lambda a_{2n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \end{pmatrix}$$

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- Idea: apply elementary transformations to a matrix to bring it to the upper triangular form and keeping track of how the determinant changes on the way:

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \end{pmatrix} = \binom{\text{some elementary}}{\text{transformations}} = \det\begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \tilde{a}_{13} & \cdots & \tilde{a}_{1n} \\ 0 & \tilde{a}_{22} & \tilde{a}_{23} & \cdots & \tilde{a}_{2n} \\ 0 & 0 & \tilde{a}_{33} & \cdots & \tilde{a}_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \end{pmatrix}$$

and use that the last determinant is equal to $\prod_i \tilde{a}_{ii}$.

- Let us list important properties of the determinant:

(i) $\det A^\top = \det A$

(ii) $\det(\lambda A) = \lambda^n \det A$

(iii) $\boxed{\det(A+B) \neq \det A + \det B}$, for example

$$\det\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 4 \neq \det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(iv) $\boxed{\det(AB) = \det(A)\cdot\det(B)}$

(v) $\operatorname{Rank} A = n \iff \det A \neq 0$

- There is another (third so far!) way to compute $\det A$, it's called the cofactor or Laplace expansion.

Definition 3. *A minor M_{ij} corresponding to the elements a_{ij} of a square matrix A is the determinant of a submatrix obtained by removing from A both i^{th} column and j^{th} row:*

$$M_{ij} = \det\begin{pmatrix} a_{11} & a_{12} & & & \\ a_{21} & a_{22} & & | & \\ & & \ddots & | & \\ - & - & - & a_{ij} & - \\ & & & | & \\ & & & | & \end{pmatrix} \leftarrow i.$$
$$\uparrow j$$

The cofactor of the element a_{ij} is defined by

$$C_{ij} = (-1)^{i+j} M_{ij}.$$

Note that the factor $(-1)^{i+j}$ may be found as follows:

$$\begin{pmatrix} +1 & -1 & 1 & \dots \\ -1 & 1 & -1 & \dots \\ 1 & -1 & 1 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Theorem 1 (Laplace expansion by column)**.** *For any choice of column j, we have*

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} = \sum_{k=1}^{n} a_{kj}C_{kj}.$$

Theorem 2 (Laplace expansion by row)**.** *For any choice of row i, we have*

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} = \sum_{k=1}^{n} a_{ik}C_{ik}.$$