Mathématiques II
HEC Lausanne, Spring 2025

Theory sheet 6

Definition of a quadratic form

Definition 1. *Let $A \in M_{n,n}$ be square matrix. A quadratic form on $\mathbb{R}^n$ is a function $f : \mathbb{R}^n \to \mathbb{R}$ of the form*

$$f(\boldsymbol{x}) = \sum_{i,j=1}^{n} a_{ij} x_i x_j.$$

Alternatively, we can write

$$f(\boldsymbol{x}) = \boldsymbol{x}^\top A \boldsymbol{x}.$$

For example, if $n = 2$, we have

$$f(x_1, x_2) = a_{11}x_1^2 + a_{12}x_1x_2 + a_{21}x_2x_1 + a_{22}x_2^2.$$

If A is symmetric, we have $a_{12} = a_{21}$ and the quadratic form can be written as

$$f(x_1, x_2) = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2.$$

A few remarks:

- If A is antisymmetric ($A^\top = -A$), then

  $$\boldsymbol{x}^\top A \boldsymbol{x} = \boldsymbol{x}^\top A^\top \boldsymbol{x} = -\boldsymbol{x}^\top A \boldsymbol{x}$$

  and hence $\boldsymbol{x}^\top A \boldsymbol{x} = 0$ for all $\boldsymbol{x}$. This means that quadratic forms with antisymmetric matrices are equal to zero.

- Since any matrix can be decomposed into a symmetric and an antisymmetric

  $$A = \frac{A + A^\top}{2} + \frac{A - A^\top}{2},$$

  we see that

  $$\boldsymbol{x}^\top A \boldsymbol{x} = \boldsymbol{x}^\top \frac{A + A^\top}{2} \boldsymbol{x} + \boldsymbol{x}^\top \frac{A - A^\top}{2} \boldsymbol{x} = \boldsymbol{x}^\top \frac{A + A^\top}{2} \boldsymbol{x},$$

  because the second term gives zero.

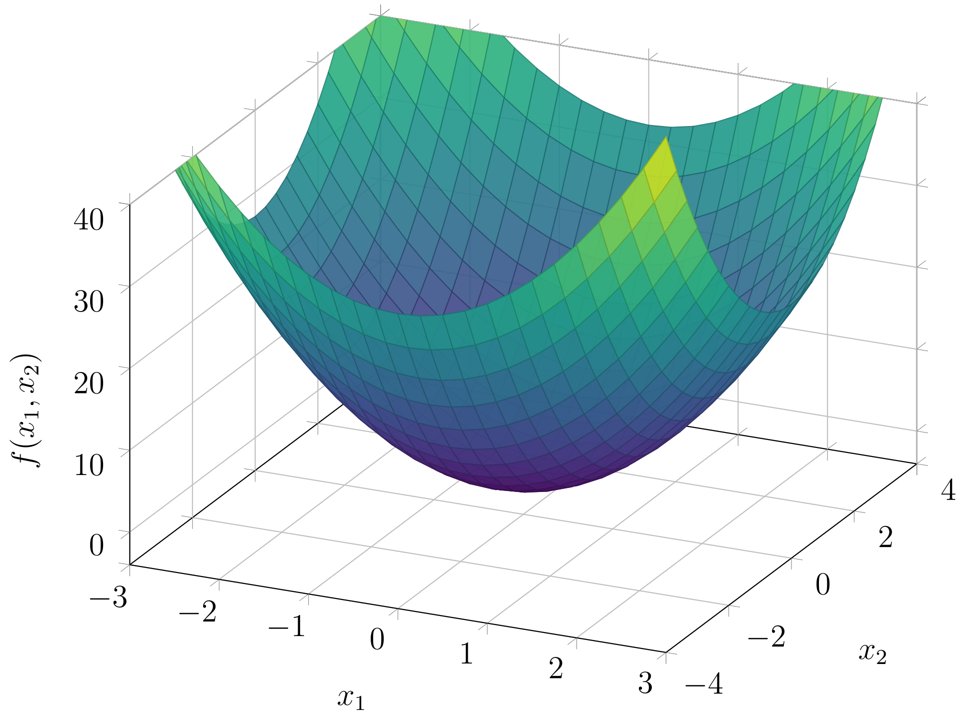
- Hence, we can always assume that A is symmetric. Its antisymmetric part disappears from the quadratic form anyway!

- From now on we always assume that $A = A^\top$.

Graphs of quadratic forms have parabolic or hyperbolic shapes. For example, the graph of

$$f(x_1, x_2) = 3x_1^2 + 2x_2^2$$

is a paraboloid. The graph of this function is shown in the following figure:



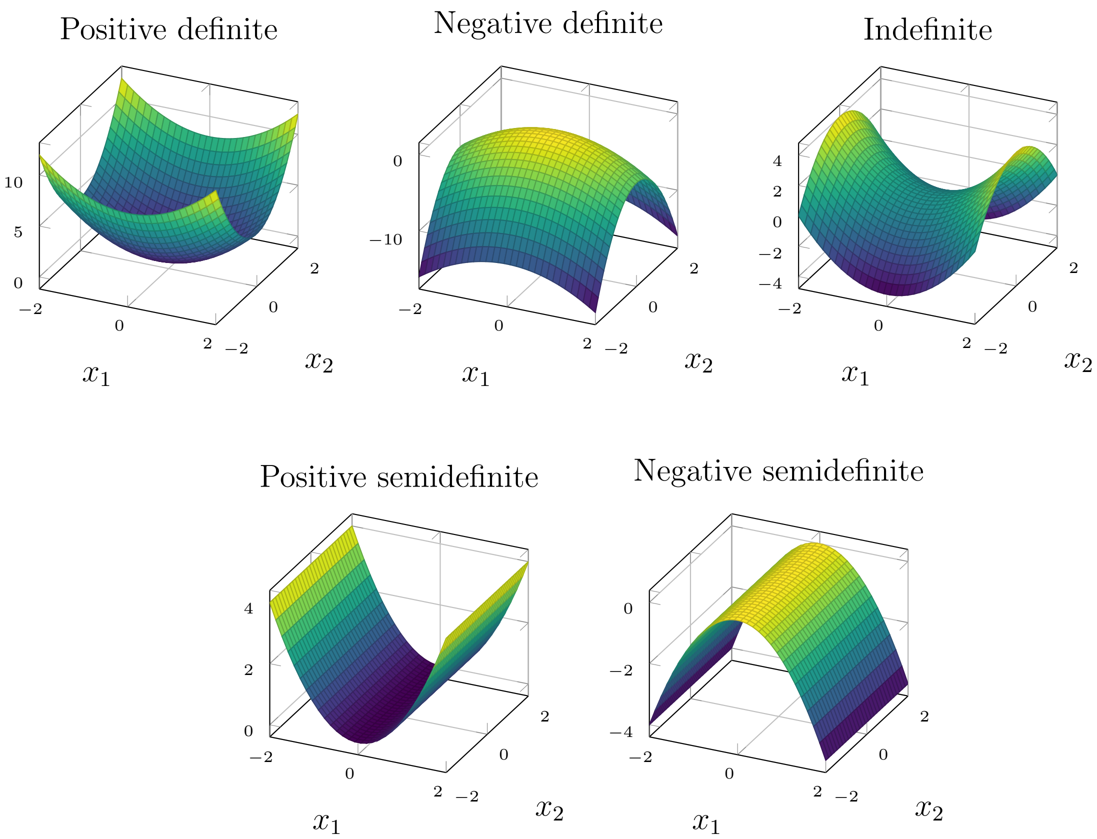
In the following section we shall describe all possible shapes these graphs can take.

Classification of quadratic forms

Definition 2. *A quadratic form* $f(\boldsymbol{x}) = \boldsymbol{x}^\top A\boldsymbol{x}$ *is called*

- *positive definite if* $f(\boldsymbol{x}) > 0$ *for all* $\boldsymbol{x}$*. In this case the graph is a paraboloid opening upwards.*
- *negative definite if* $f(\boldsymbol{x}) < 0$ *for all* $\boldsymbol{x}$*. In in this case the graph is a paraboloid opening downwards.*
- *indefinite if* $f(\boldsymbol{x})$ *takes both positive and negative values. In this case the graph is a hyperboloid.*
- *positive semidefinite if* $f(\boldsymbol{x}) \geq 0$ *(not strictly) for all* $\boldsymbol{x}$*.*
- *negative semidefinite if* $f(\boldsymbol{x}) \leq 0$ *for all* $\boldsymbol{x}$*.*

In these cases, we shall say that f *is of positive definite/negative definite/ indefinite/positive semidefinite/negative semidefinite type.*



The examples shown in these pictures are:

- **Positive definite**:
$$f(x_1, x_2) = x_1^2 + 2x_2^2 = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$
- **Negative definite:**
$$f(x_1, x_2) = -x_1^2 - 3x_2^2 = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$
- **Indefinite:**
$$f(x_1, x_2) = x_1^2 - x_2^2 = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$
- **Positive semidefinite:**
$$f(x_1, x_2) = x_1^2 = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$
- **Negative semidefinite:**
$$f(x_1, x_2) = -x_1^2 = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Case of diagonal matrices

If A is diagonal ($a_{ij} = 0$ for $i \neq j$), then the quadratic form can be written as

$$f(\boldsymbol{x}) = \sum_{i=1}^{n} a_{ii} x_i^2.$$

Hence,

- f is positive definite if $a_{ii} > 0$ for all i .
- f is negative definite if $a_{ii} < 0$ for all i .
- f is indefinite if $a_{ii} > 0$ for some i and $a_{ii} < 0$ for some i .
- f is positive semidefinite if $a_{ii} \geq 0$ for all i and $a_{ii} > 0$ for at least one i .
- f is negative semidefinite if $a_{ii} \leq 0$ for all i and $a_{ii} < 0$ for at least one i .

Hence, for diagonal matrices it is very easy to check the type of the quadratic form. We shall see later on that the general case is not much more complicated, but the answer depends on the eigenvalues of the matrix A instead of its diagonal elements.

Sylvester's law of inertia

One easy way to check whether the type of a quadratic form f is to rewrite it as sum of squares with different coefficients:

$$f(x_1, x_2) = 3x_1^2 + 12x_1x_2 + 5x_2^2 = 3(x_1 + 2x_2)^2 - 7x_2^2.$$

In this form it is easy to see that f is indefinite, because it contains a positive and a negative term.

Finding such representation is done by completing the square:

$$3x_1^2 + 12x_1x_2$$

is almost a perfect square. We can add and subtract $12x_2^2$ to obtain

$$\underbrace{3x_1^2 + 12x_1x_2 + 12x_2^2}_{=3(x_1+2x_2)^2} - 12x_2^2 + 5x_2^2 = 3(x_1 + 2x_2)^2 - 7x_2^2.$$

If we apply the same procedure to a general quadratic form in $\mathbb{R}^2$, we will discover something interesting:

$$\begin{aligned}
f(x_1, x_2) &= a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2 \\
&= a_{11}\left(x_1^2 + \frac{2a_{12}}{a_{11}}x_1x_2 + \frac{a_{22}}{a_{11}}x_2^2\right) \\
&= a_{11}\left(x_1 + \frac{a_{12}}{a_{11}}x_2\right)^2 + \left(a_{22} - \frac{a_{12}^2}{a_{11}}\right)x_2^2 \\
&= a_{11}(\dots)^2 + \frac{a_{22}a_{11} - a_{12}^2}{a_{11}}x_2^2 \\
&= a_{11}(\dots)^2 + \frac{\det\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22}\end{pmatrix}}{a_{11}}x_2^2
\end{aligned}$$

Here we replaced the expression in the first parenthesis by $(\dots)^2$ because it is not important for determination of the type.

Is the appearance of the determinant a coincidence? No, it is not. In fact, if we repeat the same calculation for a quadratic form

$$f(\boldsymbol{x}) = \boldsymbol{x}^\top A\boldsymbol{x}, \quad A \in M_{3,3}(\mathbb{R}),$$

we will find that

$$f(x_1, x_2, x_3) = a_{11}(\dots)^2 + \frac{\det\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22}\end{pmatrix}}{a_{11}}(\dots)^2 + \frac{\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33}\end{pmatrix}}{\det\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22}\end{pmatrix}}(\dots)^2.$$

The coefficients of the squares are the ratios of principal minors of the matrix A. Since we have just seen that the type of f is determined by signs of these coefficients, we can conclude that the type of a quadratic form is determined by the signs of the principal minors of the matrix A. This is the content of the Sylvester's law of inertia.

Theorem 1 (Sylvester's law of inertia)**.** *A quadratic form $f(\boldsymbol{x}) = \boldsymbol{x}^\top A\boldsymbol{x}$ is*

- *positive definite if and only if all principal minors of A are positive:*

$$a_{11} > 0, \quad \det\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22}\end{pmatrix} > 0, \quad \det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33}\end{pmatrix} > 0, \quad \dots$$

- *negative definite if and only if all principal minors of A have alternating signs*

$$a_{11} < 0, \quad \det\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22}\end{pmatrix} > 0, \quad \det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33}\end{pmatrix} < 0, \quad \dots$$

- *indefinite if and only if the signs of the principal minors of A are neither positive, not alternating.*
- *positive semidefinite if and only if all principal minors are non-negative, but may be zero.*

This theorem allows us to easily determine the type of a quadratic form by computing the determinants of the principal minors of the matrix A and checking their signs.

Example 1. *Let $A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$. The principal minors are*

$$a_{11} = 1 > 0, \quad \det \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} = 1 \cdot 3 - 2 \cdot 2 = -1 < 0.$$

Hence, the quadratic form $f(x_1, x_2) = x_1^2 + 4x_1x_2 + 3x_2^2$ is negative definite.

Quadratic forms in terms of eigenvalues

We have seen above that the type of a diagonal matrix is determined by the signs of its diagonal elements. It turns out that the type of a general matrix is determined by the signs of its eigenvalues. Let us first show this on a 2×2 example.

If $A \in M_{2,2}$ has two distinct eigenvalues λ_1 and λ_2, let $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$ be their corresponding eigenvectors. Then we can express any vector $\boldsymbol{x}$ in terms of $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$ as

$$\boldsymbol{x} = c_1\boldsymbol{y}_1 + c_2\boldsymbol{y}_2$$

with some coefficients c_1 and c_2. Then we have

$$\begin{aligned} \boldsymbol{x}^\top A\boldsymbol{x} &= (c_1\boldsymbol{y}_1 + c_2\boldsymbol{y}_2)^\top A(c_1\boldsymbol{y}_1 + c_2\boldsymbol{y}_2) \\ &= c_1^2\boldsymbol{y}_1^\top A\boldsymbol{y}_1 + c_2^2\boldsymbol{y}_2^\top A\boldsymbol{y}_2 + c_1c_2\boldsymbol{y}_1^\top A\boldsymbol{y}_2 + c_2c_1\boldsymbol{y}_2^\top A\boldsymbol{y}_1. \end{aligned}$$

Since A is symmetric, its eigenvectors corresponding to different eigenvalues are orthogonal

$$\boldsymbol{y}_1^\top A\boldsymbol{y}_2 = \boldsymbol{y}_2^\top A\boldsymbol{y}_1 = 0$$

(see previous lecture). For the first two terms we have

$$\boldsymbol{y}_i^\top A\boldsymbol{y}_i = \lambda_i\boldsymbol{y}_i^\top\boldsymbol{y}_i = \lambda_i|\boldsymbol{y}_i|^2, \quad i = 1, 2,$$

where $|\boldsymbol{y}|^2 = \boldsymbol{y}^\top\boldsymbol{y} = \sum y_i^2 \geq 0$ is the length or norm of $\boldsymbol{y}$. Therefore,

$$\boldsymbol{x}^\top A\boldsymbol{x} = \lambda_1 c_1^2|\boldsymbol{y}_1|^2 + \lambda_2 c_2^2|\boldsymbol{y}_2|^2$$

and its sign is determined by the signs of λ_1 and λ_2 (because everything else is positive).

Hence, in this case we see that the type of the quadratic form is determined by the signs of its eigenvalues. The same is true for general matrices, which is the content of the following theorem (also without proof):

Theorem 2. *Let $A \in M_{n,n}(\mathbb{R})$ be a symmetric matrix. The quadratic form $f(\boldsymbol{x}) = \boldsymbol{x}^\top A\boldsymbol{x}$ is*

- *positive definite if all eigenvalues of A are positive.*
- *negative definite if all eigenvalues of A are negative.*
- *indefinite if A has both positive and negative eigenvalues.*
- *positive semidefinite if all eigenvalues of A are non-negative and at least one is positive.*
- *negative semidefinite if all eigenvalues of A are non-positive and at least one is negative.*

How to determine the type of a quadratic form?

We have two options:

- Compute the principal minors of the matrix A and check their signs using Sylvester's law of inertia.
- Compute the eigenvalues of the matrix A and check their signs using the above theorem.