# Theory sheet 8

## Leontiev model

- The economy of a country is broken down into $n$ sectors/regions/cities.
- We fix the period of time (e.g. one year, one month, etc.) and consider the production of goods and consumption of each sector.
- The sells of $i^{\text{th}}$ sector to the $j^{\text{th}}$ sector is denoted by $x_{ij}$ .
- The consumption of the $i^{\text{th}}$ sector is denoted by $x_{ii}$ .
- The import of the $i^{\text{th}}$ sector is denoted by $y_i$ .
- The export of the $i^{\text{th}}$ sector is denoted by $z_i$ .
- The total production of the $i^{\text{th}}$ sector is denoted by $x_i$ .

Clearly, we have the following relations:

$$x_i = \sum_{j=1}^{n} x_{ij} + z_i.$$

**Definition 1** (Assumptions of the Leontiev model)**.**

***Assumption 1.*** *$x_{ki}$ is proportional to the production of the $i^{th}$ sector, i.e. $x_{ki} = a_{ki}x_i$ where $a_{ki}$ is the trade or exchange coefficient if $k \neq i$ and self-consumption coefficient if $k = i$ .*

***Assumption 2.*** *$y_i$ is proportional to $x_i$ , i.e. $y_i = d_i x_i$ where $d_i$ is the import coefficient.*

We can write the Leontiev model in matrix form as follows:

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}$$

or just

$$\boldsymbol{x} = A\boldsymbol{x} + \boldsymbol{z}.$$

Here $A$ is the matrix of trade coefficients, $\boldsymbol{x}$ is the vector of production and $\boldsymbol{z}$ is the vector of exports.

Note that $\boldsymbol{x}$ appears on both sides of the equation. We can solve for $\boldsymbol{x}$ by rearranging the equation:

$$\boldsymbol{x} - A\boldsymbol{x} = \boldsymbol{z} \implies (I - A)\boldsymbol{x} = \boldsymbol{z} \implies \boxed{\boldsymbol{x} = (I - A)^{-1}\boldsymbol{z}.}$$

Note that the last equation is only valid if $I - A$ is invertible. In this model, invertibility of $I - A$ has a very natural interpretation. It means that there is a unique solution for the production of each sector given the exports of each sector, that is, the production of each sector can be determined uniquely from the exports of each sector. If $I - A$ is not invertible, then there are infinitely many solutions for the production of each sector given the exports of each sector.

## Leontief Model: Example

A city has 3 main industries:

- a coal mine,
- a power plant, and
- a local railway.

To produce $1 worth of coal, the mine must purchase:

- $0.25 worth of electricity and
- $0.25 worth of rail transport.

To produce $1 worth of electricity, the power plant must use:

- $0.65 worth of coal,
- $0.05 worth of its own electricity, and
- $0.05 worth of rail transport.

To provide $1 worth of transport, the railway company must use:

- $0.55 worth of coal,
- $0.10 worth of electricity.

During a certain week, the mine receives an order from outside the city for $50,000 worth of coal, and the power plant receives an order for $25,000 worth of electricity, also from outside. There is no external demand for the local railway.

What quantities must these three industries produce during this week to satisfy their own demand and the external demand?

### Solution:

For the week in question, let

$x_1 =$ the value of the total production of the mine

$x_2 =$ the total value of electricity production

$x_3 =$ the total value of railway service.

The exchange/consumption matrix:

$$A = \begin{pmatrix} 0 & 0.65 & 0.55 \\ 0.25 & 0.05 & 0.10 \\ 0.25 & 0.05 & 0 \end{pmatrix}$$

Therefore, the equation $(I - A) \cdot \boldsymbol{x} = \boldsymbol{z}$ reads

$$\begin{pmatrix} 1.00 & -0.65 & -0.55 \\ -0.25 & 0.95 & -0.10 \\ -0.25 & -0.05 & 1.00 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 50,000 \\ 25,000 \\ 0 \end{pmatrix}$$

and therefore

$$\boldsymbol{x} = (I - A)^{-1}\boldsymbol{z} = \frac{1}{503} \begin{pmatrix} 756 & 542 & 470 \\ 220 & 690 & 190 \\ 200 & 170 & 630 \end{pmatrix} \begin{pmatrix} 50,000 \\ 25,000 \\ 0 \end{pmatrix} = \begin{pmatrix} 102,087 \\ 56,163 \\ 28,330 \end{pmatrix}.$$

Interpretation: in order to export \$50,000 worth of coal and \$25,000 worth of electricity, the mine must produce \$102,087 worth of coal, the power plant must produce \$56,163 worth of electricity, and the railway must produce \$28,330 worth of railway service.

## Viability of a sector

To be viable, a sector must be able to produce enough goods to satisfy its own consumption and the consumption of other sectors:

$$x_{1i} + x_{2i} + \cdots + x_{ni} + y_i < x_i.$$

Plugging in the definitions of $y_i$ and $x_{ij}$, we get

$$\sum_{j=1}^{n} a_{ji}x_i + d_i x_i < x_i.$$

Note that the sum is over $j$ and not $i$, so we can extract $x_i$ from it. Dividing both sides by $x_i$, we get

$$\sum_{j=1}^{n} a_{ji} + d_i < 1.$$

This is the viability condition for the $i^{\text{th}}$ sector.

# Matrix notation for imports

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

or just

$$\boldsymbol{y} = D\boldsymbol{x},$$

where $D$ is the diagonal matrix of import coefficients.

# Trade balance

The trade balance is the difference between the total exports and the total imports of a sector:

$$B = \sum_{i=1}^{n} z_i - \sum_{i=1}^{n} y_i.$$

Is is convenient to express the trade balance in matrix form:

$$B = \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} z_1 - y_1 \\ z_2 - y_2 \\ \vdots \\ z_n - y_n \end{pmatrix} = S^\top(\boldsymbol{z} - \boldsymbol{y}), \quad \text{where} \quad S = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Now we can plug $\boldsymbol{y} = D\boldsymbol{x}$ and obtain

$$\begin{aligned} B &= S^\top(\boldsymbol{z} - D\boldsymbol{x}) \\ &= S^\top(\boldsymbol{z} - D(I - A)^{-1}\boldsymbol{z}) && \text{because } \boldsymbol{x} = (I - A)^{-1}\boldsymbol{z} \\ &= S^\top(I - D(I - A)^{-1})\boldsymbol{z}. \end{aligned}$$

Final formula for trade balance:

$$B = S^\top(I - D(I - A)^{-1})\boldsymbol{z}.$$

# Another example

Consider an economy with 3 sectors: K, L, M. Let the imports and exports be

$$\text{Imports } \boldsymbol{y} = \begin{pmatrix} 300 \\ 500 \\ 150 \end{pmatrix} \quad \text{Exports } \boldsymbol{z} = \begin{pmatrix} 250 \\ 1200 \\ 300 \end{pmatrix}$$

Given that $d_1 = 30\%$, $d_2 = 25\%$, $d_3 = 10\%$, what is the production $\boldsymbol{x}$ of each sector? First, we need to find the import matrix $D$:

$$D = \begin{pmatrix} 0.3 & 0 & 0 \\ 0 & 0.25 & 0 \\ 0 & 0 & 0.1 \end{pmatrix}.$$

Then we can express the imports in terms of the production:

$$\boldsymbol{y} = D\boldsymbol{x} \implies X = D^{-1}Y$$

It remains to find $D^{-1}$ and multiply it by $\boldsymbol{y}$:

$$D^{-1} = \begin{pmatrix} \frac{1}{0.3} & 0 & 0 \\ 0 & \frac{1}{0.25} & 0 \\ 0 & 0 & \frac{1}{0.1} \end{pmatrix} \implies \boldsymbol{x} = \begin{pmatrix} \frac{1}{0.3} & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 10 \end{pmatrix} \begin{pmatrix} 300 \\ 500 \\ 150 \end{pmatrix} = \begin{pmatrix} 1000 \\ 2000 \\ 1500 \end{pmatrix}.$$

Interpretation: the production of sector K is \$1000, the production of sector L is \$2000, and the production of sector M is \$1500.

Assuming that

- There is no self-consumption,
- No exchanges between K and M,
- Sector M doesn't buy anything from sector L,

find the exchange matrix $A$ and the trade balance. By the assumptions, we can write the exchange matrix as follows:

$$A = \begin{pmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ 0 & a_{32} & 0 \end{pmatrix}.$$

Now we write down the equation for the production of each sector:

$$\boldsymbol{x} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{z} \quad \text{or} \quad (I - A)\boldsymbol{x} = \boldsymbol{z}$$

and solve for $A$:

$$\begin{pmatrix} 1 & -a_{12} & 0 \\ -a_{21} & 1 & 0 \\ 0 & -a_{32} & 1 \end{pmatrix} \begin{pmatrix} 1000 \\ 2000 \\ 1500 \end{pmatrix} = \begin{pmatrix} 250 \\ 1200 \\ 300 \end{pmatrix} \implies \begin{cases} a_{12} = 0.375 \\ a_{21} = 0.8 \\ a_{32} = 0.6 \end{cases}$$

The trade balance is given by

$$B = \sum_{i=1}^{n} z_i - \sum_{i=1}^{n} y_i = 250 + 1200 + 300 - (300 + 500 + 150) = 800.$$

The trade balance is positive, which means that the economy is exporting more than it is importing.