UNIL | Université de Lausanne

FACULTÉ DES HAUTES ÉTUDES COMMERCIALES

DÉPARTEMENT DE SCIENCES ACTUARIELLES

Towards a theory of multivariate Gaussian extremes

THÈSE DE DOCTORAT

présentée à la

Faculté des Hautes Études Commerciales
de l'Université de Lausanne

pour l'obtention du grade de

Doctorat en sciences actuarielles

par

Pavel IEVLEV

Directeur de thèse
Prof. Enkelejd Hashorva

Jury
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University of Lausanne
Faculty of Business and Economics

PhD in Actuarial Science

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Pavel IEVLEV

and have found it to meet the requirements for a doctoral thesis.
All revisions that I or committee members
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have been addressed to my entire satisfaction.

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Acknowledgements

I did not have a proper acknowledgements section in my first PhD thesis, so I have decided to use this section as an opportunity to say what I ought to have said to the close few who have been around me throughout the years.

First and foremost, I want to thank my supervisor, Enkelejd Hashorva. Thank you for being infinitely patient and bearing with my snailish pace of work; for being ever-supportive, but never pushing; for our long and passionate conversations ranging from elevated to earthbound, but never mundane. I am deeply grateful for everything you did for me, personally and professionally. I feel incredibly lucky to have you in my corner.

I owe special thanks to Krzysztof Dębicki, who wasn't technically my co-advisor, but from whom I learnt so much of our craft. I appreciate my visits to Wrocław and the work we did together. I hope this colaboration is far from being finished.

Thanks to my senior collegues Nikolai Kriukov, Gregory Jasnovidov and Konrad Krystecki for the warm welcome when I first came to Lausanne and the many mathematical conversations we've had together. To my junior collegues Svyatoslav Novikov and Timofei Shashkov, with whom we have shared Extranef 134 for the last two years and made so many discoveries by that whiteboard, thank you for always being up for a math challenge and all the work we did (and will continue to do) together.

To my -friends Ilnur Baibulov, Ivan Tambovtcev and Ekaterina Zlobina, whose presence in my life, after all these years, I dare not take for granted. I owe you all so much it would be futile to even attempt a list, but I want you to know that being around you remains one of my most cherished joys.

To my outside-of-academia friends Vitalii Diakun and Elizaveta Isaicheva, with whom we've shared many beers, but much more memories. I thank you for being who you are and the evergreen sensation of being close by even if you're four thousand kilometers away. Even though life has scattered us across the globe, I hold tight onto the hope to meet you in Saint Petersburg again. When I'm feeling low, I simply conjure the thought of the last New Year's Eve we've spent together in my appartment on 6-ya Sovetskaya, and this thought alone keeps me going.

To my mother Marina Polubareva-Engberg, whose support and seemingly infinite energy under any circumstances, whose protectiveness and determination, whose selfless love have shaped me into what I am. Thank you for pushing me when I need a push the most, and for being both a mother and a true friend. To my father Nikolai Ievlev, without whom I would not have chosen this path. You have supported me and made it all possible in many different ways, big and small, and I'm eternally grateful for this. It would be impossible to list all I owe you, so I'll simply thank you for showing me how fun physics is (thus launching my long journey). To my grandmother, Natalia Polubareva, and my grandfather, Alexey Polubarev, whose wisdom, resilience, and unwavering belief in me have been a source of strength and inspiration throughout my life.

Finally, I want to thank Anna Smirnova for being by my side for these seven years. I know it has not been easy, but you made it all worth it. Thank you for having faith in me, even when I struggle to find it in myself.

Thank you all, this thesis is as much yours as it is mine.

Summary

This thesis explores several directions in the theory of extremal behaviour of Gaussian processes opened by a recent paper by K. Dębicki, E. Hashorva and L. Wang (2019). In Chapter 2 we extend their results from processes to a simple yet rich class of non-homogenous vector-valued Gaussian random fields. As an application of this extension, we derive exact asymptotic approximations of the so-called double crossing probabilities. In Chapters 3, we present a new class of covariance matrix functions of exponential type, which we later apply in Chapter 4 in conjunction with the Gordon inequality to the study of extremes of locally-homogenous Gaussian random fields. This allows to significantly simplify proofs and avoid using stringent assumptions, required by the previously available techniques. In Chapter 5, we introduce a class of multivariate Gaussian processes, Brownian decision trees, closely related to the well-known Branching Brownian motion and study their extremal behaviour. In Chapter 6, we investigate the Parisian ruin in the so-called many inputs proportional reinsurance risk model with fractional Brownian motion input.

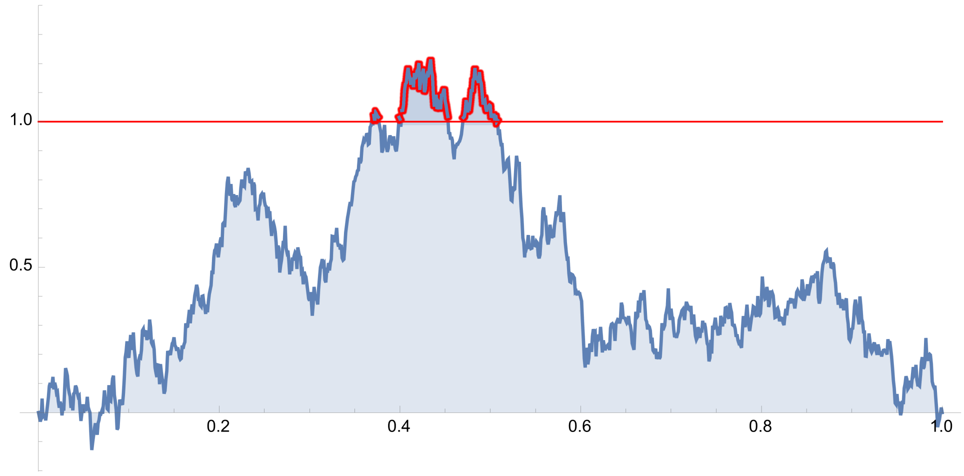
Résumé

Cette thèse explore plusieurs directions dans la théorie du comportement extrémal des processus gaussiens, ouvertes par un article récent de K. Dębicki, E. Hashorva et L. Wang (2019). Dans le Chapitre 2, nous étendons leurs résultats des processus à une classe simple mais riche de champs aléatoires gaussiens vectoriels non homogènes. Comme application de cette extension, nous dérivons des approximations asymptotiques exactes des probabilités dites de double franchissement. Dans le Chapitre 3, nous présentons une nouvelle classe de fonctions de matrice de covariance de type exponentiel, que nous appliquons par la suite dans le Chapitre 4 en conjonction avec l'inégalité de Gordon à l'étude des extrêmes des champs aléatoires gaussiens localement homogènes. Cela permet de simplifier considérablement les démonstrations et d'éviter l'utilisation d'hypothèses strictes requises par les techniques précédemment disponibles. Au Chapitre 5 nous introduisons une classe de processus gaussiens multivariés, les arbres de décision browniens, étroitement liés au bien connu mouvement brownien branchant, et nous étudions leur comportement extrême. Dans le Chapitre 6, nous étudions la ruine parisienne dans le modèle de réassurance à entrées multiples proportionnelles avec un mouvement brownien fractionnaire en entrée.

Contents

Chapter 1

Introduction



Asymptotical analysis of Gaussian fields and processes has a long and richly branched history, tracing its roots to the early part of the last century. In this thesis, we shall focus on the recent developments around seminal contributions made in the end of 1960s by J. Pickands III in two papers [1] and [2]. In these papers he formulated a general approach, known now as *the double sum method*, for deriving exact asymptotical approximations of the so-called high exceedance probabilities

$$\mathbb{P}\left\{\exists\, t \in [0,T] \colon X(t) > u\right\} \quad \text{as} \quad u \to \infty.$$

Here X is a centered Gaussian process on $[0,T]$ with values in $\mathbb{R}$ satisfying some standard assumptions. These results have since been extended in numerous directions, including

1. replacing $[0,T]$ by some subset T of $\mathbb{R}^n$, see pioneering works [3, 4, 5, 6, 6], as well recent developments [7, 8, 9, 10];
2. replacing $[0,T]$ by an asymptotically dense grid (see [11]) or even random grid (see [12]);
3. allowing X to be non-centered, see [13];
4. allowing X and T to depend on u in some weak but non-trivial way, see [14];
5. allowing X to be asymptotically Gaussian, see [15];
6. replacing the event $\{\exists\, t \colon X(t) > u\} = \{\sup_t X(t) > u\}$ by a more general class of events of the form $\{\Gamma(X - u)\}$ with Γ a functional of the entire path, see, for example, [16, 17] and a general contribution [18].

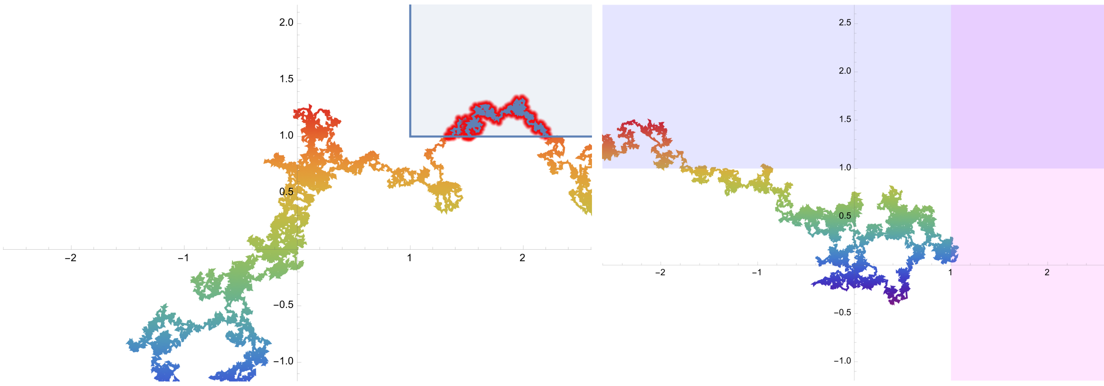
Despite the developments mentioned above, up until recently little has been known about exact asymptotics of Gaussian high exceedance probabilities *in the multivariate case*. A deep contribution [19] has paved a way towards various problems of the following kind:

$$\mathbb{P}\left\{\exists\, t \in [0, T] \colon \boldsymbol{X}(t) > u\boldsymbol{b}\right\} \quad \text{as} \quad u \to \infty$$

for $\boldsymbol{b} \in \mathbb{R}^d \setminus (-\infty, 0]^d$ and $\boldsymbol{X}$ being a continuous centered Gaussian process. Here "$>$" denotes the componentwise (Hadamard) comparison. As it turns out, these problems are much more challenging than the univariate ones due to the lack of several techniques, such as the Slepian and the Borell-TIS inequalities, which are crucial for the univariate case. The reader can find the detailed account of this shortage in the introduction to the aforementioned paper.

Development of the multivariate Gaussian extremes theory has opened the door to a wide range of new problems absent in the univariate case. Here are some notable possibilities:

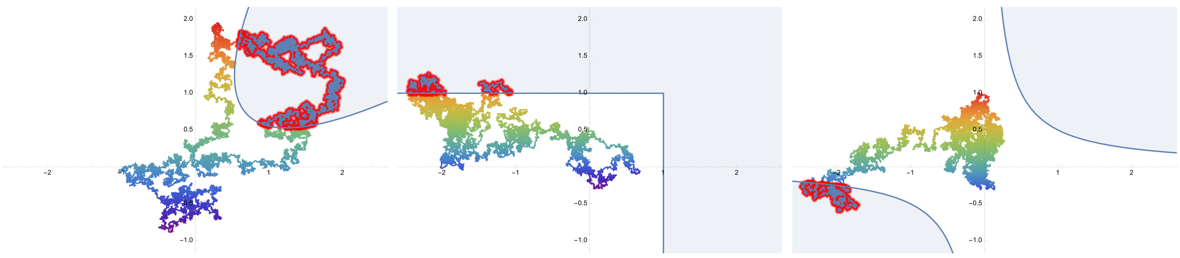
1. The components of $\boldsymbol{X}$ may exceed a certain threshold simultaneously, which corresponds to the event $\{\exists\, t \colon X_1(t) > b_1 u,\ X_2(t) > b_2 u\}$ (see [20]), or non-simultaneously, which corresponds to another event $\{\exists\, t_1, t_2 \colon X_1(t_1) > b_1 u,\ X_2(t_2) > b_2 u\}$ (see [21]);



(a) Simultaneous exceedance. (b) Non-simultaneous exceedance.

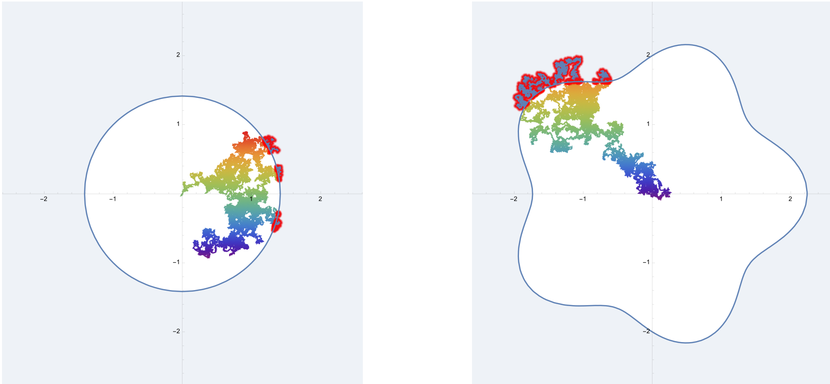
2. The exceedance event may be more generally defined as $\{\exists\, t \colon \boldsymbol{X}(t) \in B_u\}$, where B_u is a parametric family of sets escaping to infinity in some way as $u \to \infty$ (see an upcoming paper by K. Dębicki, N. Kriukov and S. Novikov);

3. Particular cases of the previous point include the high exceedances of the i-th order statistic process $X_{i:n}(t)$ (see [22]), the product $\prod_{i=1}^n X_i(t)$ (see [23]), the norm $\|\boldsymbol{X}(t)\|$ (see [24]) or other functionals which break Gaussianity but have simple geometric interpretation;



(c) Exceedance from a parabolic region. (d) Order statistics exceedance. (e) Product exceedance.

4. Many new problems arise from non-trivial dependence structures between the components of vector $\boldsymbol{X}$.



(f) Exceedance of the norm $\|\boldsymbol{X}(t)\|$. (g) $\{\exists\, t\colon \boldsymbol{X}(t) \in \{(r,\theta)\colon r \geq u(4+\cos 5\theta)\}\}$

Two more examples for points 2 and 3.

One could say that a theory is complete if most of its problems can be solved by a tedious application of known tools. In this sense, theory of multivariate Gaussian extremes is far from complete: many open questions are waiting to be solved, and we believe that some new tools needed to tackle them are yet to be invented. By this thesis, we hope to make a small step towards such completion.

In Chapter 2, we extend the results of [19] to a simple yet rich class of non-homogenous vector valued Gaussian *fields*, that is, Gaussian random functions $\boldsymbol{X}\colon \mathbb{R}^n \to \mathbb{R}^d$. As it turns out, the vector and the field structures intertwine non-trivially. As an application of general theorem presented in this chapter, we present several results on the so-called double crossing probabilities: the probability that a real-valued process first hits a high positive barrier and then a low negative barrier within a finite time horizon (see also [25]).

To introduce Chapters 3 and 4, let us expand on what we said above about Slepian inequality being missing from a multivariate Gaussian extremes toolbox. This inequality plays a crucial role in the univariate double sum method allowing to find for a given process X, numer $\varepsilon > 0$ and a short interval a pair of processes $Y_{\pm,\varepsilon}$, which stochastically dominate X from above and from below and are close to X as $\varepsilon \to 0$ on this interval. This in turn allows one to avoid uniformity issues (see [19] for details), which otherwise require heavier proofs and more stringent assumptions. A multivariate extension of Slepian inequality, known as the Gordon inequality, was previously believed to be inaccessible due to a lack of reference processes $\boldsymbol{Y}_{\pm,\varepsilon}$. In Chapter 3, we introduce a new class of matrix-valued covariance functions. This class, interesting in its own right, we later apply in Chapter 4 in conjunction with the Gordon inequality to the study of the extremes of locally-homogenous Gaussian random fields.

In Chapter 5, we investigate high excursions of a new process, which we termed *the Brownian decision tree*. This process is a close relative of the standard branching Brownian motion and it may be informally described as follows: at time $t = 0$, a Brownian motion B sets off from zero and runs freely until a non-random time $\tau_1 > 0$, at which it splits into $N_1 \geq 1$ conditionally on the common past independent Brownian motions. The resulting vector-valued process again runs freely up to some time point $\tau_2 > \tau_1$, where each of its components splits again into $N_2 \geq 1$ particles, and the construction recursively repeats. There are two differences between this and the classical BBm model as presented, for example, in the seminal paper by Bramson [26]. Firstly, the branching times are non-random, whereas in the standard model the distances between them are exponentially distributed. Secondly, all branches (that is, the components of the vector-valued process described above) undergo splitting into the same number of offsprings and at the same time (in the classical BBm model each branch has its own branching clock).

This, along with the usual description of the classical BBm model as a process indexed by a tree, suggests the name *Brownian decision trees*, where the word "decision" refers to the specific type of trees branching at the same points and into the same amount of branches.

In Chapter 6, we investigate the Parisian ruin probability for a class of Gaussian processes with power-asymmetric behaviour of the variance near the unique optimal point. This result is then applied to the study of Parisian ruin in the so-called many-inputs proportional reinsurance risk model with fractional Brownian motion input.

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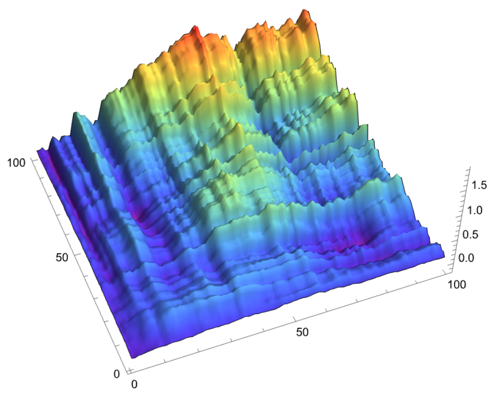
Chapter 2

Extremes of vector-valued locally additive Gaussian fields with application to double crossing probabilities



The asymptotic analysis of high exceedance probabilities for Gaussian processes and fields has been a blooming research area since J. Pickands introduced the now-standard techniques in the late 60's. The *vector-valued* processes, however, have long remained out of reach due to the lack of some key tools including Slepian's lemma, Borell-TIS and Piterbarg inequalities. In a 2020 paper by K. Dębicki, E. Hashorva and L. Wang, the authors extended the double-sum method to a large class of vector-valued processes, both stationary and non-stationary. In this contribution we make one step forward, extending these results to a simple yet rich class of non-homogenous vector-valued Gaussian *fields*. As an application of our findings, we present an exact asymptotic result for the probability that a real-valued process first hits a high positive barrier and then a low negative barrier within a finite time horizon.

This is a joint work with N. Kriukov, resubmitted to Electronic Journal of Probability.

2.1 Introduction

The asymptotic analysis of high exceedance probabilities for Gaussian processes and fields has been a blooming research area for several decades. The most classical results due to J. Pickands [1, 2] give the asymptotics of

$$\mathbb{P}\left\{\exists\, t \in [0,T] \colon X(t) > u\right\}, \quad \text{as} \quad u \to \infty,$$

when X is a centered Gaussian process on $[0,T]$ with values in $\mathbb{R}$ satisfying some standard assumptions. These results have since been extended in numerous directions. Among these, little was known about similar problems for vector-valued processes until very recently. A deep contribution [3] paved a way to the asymptotic analysis of the probabilities

$$\mathbb{P}\left\{\exists\, t \in [0,T]\ \forall\, i \in \{1,\ldots,d\} \colon X_i(t) > u\, b_i\right\}, \quad \text{as} \quad u \to \infty,$$

for a centered $\mathbb{R}^d$-valued Gaussian process $\boldsymbol{X}(t)$, $t \in [0,T]$ and $\boldsymbol{b} \in \mathbb{R}^d$ with at least one positive component. As the authors point out, even the seemingly trivial case of centered $\boldsymbol{X}$ with independent components is quite challenging (see [4, 5, 6]). See also the earlier studies [7, 8, 9]. We want to mention in passing the non-centered vector-valued high exceedance problems, which were initially studied for linear transformations of $\mathbb{R}^d$-valued Brownian motion in [10, 11] and have recently been extended to linear transformations of stationary increments processes with independent components satisfying Berman condition in [12]. The approximations of high exceedance probabilities of vector-valued processes appear naturally in various applications including statistics, ruin theory and queueing theory, see e.g., [5, 13, 14, 15, 16]

Another direction in which the classical theorems may be extended involves high exceedances of Gaussian random fields $X(\boldsymbol{t})$, $\boldsymbol{t} \in E \subset \mathbb{R}^n$. Deep results of this type are known since at least the 70's and some of them are presented in the well-known monograph by Piterbarg [17]. See [18] and [19] for some recent developments.

In the current contribution we prove several results related to the aforementioned generalizations, that is, to centered Gaussian random vector fields $\boldsymbol{X}(\boldsymbol{t})$, $\boldsymbol{t} \in \mathbb{R}^n$ taking values in $\mathbb{R}^d$ under some simplifying assumptions. More specifically, the vector fields we consider behave near the most likely point of high exceedance as sums of independent vector fields, each of which depends on one coordinate of $\boldsymbol{t}$, namely,

$$\boldsymbol{X}(\boldsymbol{t}) \approx \boldsymbol{X}_1(t_1) + \boldsymbol{X}_2(t_2) + \cdots + \boldsymbol{X}_n(t_n), \qquad \boldsymbol{t} \approx \boldsymbol{t}_*.$$

The exact meaning of "$\approx$" is described by Assumption A2 below. In developing these assumptions, our aim was to find the simplest yet fecund extension of the paper [3] to the case of miltidimensional parameter. There are two recent papers we want to mention in this regard. In [20] and [21] the so-called non-simultaneous ruin probability of a pair of correlated Brownian motions with linear trends

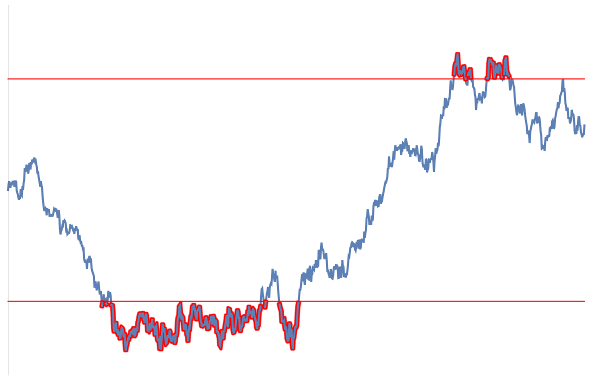
$$\mathbb{P}\left\{\exists\, (t,s) \in [0,T]^2 \colon B_1(t) - \mu_1 t > u,\ B_2(s) - \mu_2 s > u\right\}$$

was studied in the infinite horizon case $T = \infty$ and finite horizon case $T < \infty$ correspondingly. Although this probability may well be rewritten as a ruin of two dimensional vector field, our local additivity assumptions are not met in this setup. Hence these two papers remain out of reach of our Theorem 2.1. Similar results have been obtained in [22] for a class of $\mathbb{R}^2$-valued locally-stationary Gaussian random fields indexed by $\mathbb{R}^n$. See also [23] for recent developments in the case of smooth vector-valued Gaussian fields.

As an application of our findings, we present an asymptotic formula for the probability that a one-dimensional stationary Gaussian process $X(t)$, $t \in [0,T]$ hits two distant barriers: one above and one below its starting point. Namely, we derive a precise approximation of

$$\mathbb{P}\left\{\exists\, t,\ s \in [0,T]\colon X(t) > au,\ X(s) < -bu\right\}, \qquad a,\, b \geq 0 \tag{2.1}$$

as $u \to \infty$.



Double crossing event.

The probability in (2.1) can be conveniently rewritten in the vector notation as

$$\mathbb{P}\left\{\exists\, \boldsymbol{t} \in [0,T]^2\colon \boldsymbol{X}(\boldsymbol{t}) > u\boldsymbol{b}\right\}, \quad \boldsymbol{X}(\boldsymbol{t}) = (X(t_1),\ -X(t_2))^\top, \quad \boldsymbol{b} = (a,b)^\top.$$

If further the correlation function $\rho(t) = \mathbb{E}\left\{X(0)X(t)\right\}$ is positive, strictly smaller than 1 and satisfies

$$\rho(t) \sim 1 - \vartheta\, t^\alpha + o\left(t^\alpha\right) \quad \text{as} \quad t \to 0,$$

with some $\vartheta > 0$ and $\alpha \in (0,2]$, the problem falls within the scope of our assumptions after some minor adjustments. In view of Theorem 2.2 in Section 2.3 we obtain

$$\mathbb{P}\left\{\exists\, t,\ s \in [0,T]\colon X(t) > au,\ X(s) < -bu\right\} \sim c\, u^{\max\{0, 4/\alpha - 2\}}\, \mathbb{P}\left\{X(0) > au,\ X(T) < -bu\right\},$$

where the constant $c \in (0,\infty)$ is given in Theorem 2.1.

We now recapitulate the main ingredients of our approach and emphasize a few points which, in our opinion, are worth mentioning.

First, it is known that the high exceedance event of a vector-valued Gaussian process $\boldsymbol{X}(t)$, $t \in [0,T]$ is most likely to happen near the maximizer of the so-called generalized variance function, defined as

$$\sigma_{\boldsymbol{b}}^{-2}(t) = \min_{\boldsymbol{x} \geq \boldsymbol{b}} \boldsymbol{x}^\top\, \Sigma^{-1}(t)\, \boldsymbol{x},$$

where $\Sigma(t) = R(t,t)$ and $R(t,s) = \mathbb{E}\left\{\boldsymbol{X}(t)\, \boldsymbol{X}(s)^\top\right\}$ is the covariance matrix function of the process $\boldsymbol{X}$. A similar function with $\boldsymbol{t} \in [0,T]^n$ instead of $t \in [0,T]$ plays the same role in the case of random fields. The asymptotic analysis of the high exceedance probability usually begins with showing that the probability that the overshoot happens outside some small vicinity of this maximizer is negligible.

Secondly, in our approach to the issues arising from the dimensionality of the process, we closely follow the paper [3], but there are two important differences. To explain the first, let us briefly reproduce here Assumption **D2** of the paper. Let $R(t,s)$ be the covariance matrix of an $\mathbb{R}^d$-valued Gaussian process $X(t)$, $t \in [0,T]$ and denote $\Sigma(t) = R(t,t)$. Then, Assumption **D2** demands that for all $t \in [0,T]$ there be a continuous $d \times d$ matrix-function $A(t)$ such that

$$\Sigma(t) = A(t)\, A(t)^\top$$

and with some $d \times d$ real matrix Ξ and $\beta' > 0$ holds

$$A(t) = A(t_0) - |t - t_0|^{\beta'}\, \Xi + o\left(|t - t_0|^{\beta'}\right) \tag{2.2}$$

as $t \to t_0$, where t_0 is the point which maximizes the generalized variance $\sigma^2_{\boldsymbol{b}}(t)$. From a practical point of view, it is not always easy to compute $A(t)$ starting with $\Sigma(t)$. This is why instead of assuming something similar to **D2** and **D3** (the latter also requires knowing $A(t)$ along with its inverse), we impose assumptions directly on the asymptotic expansion of the covariance matrix $R(t, s)$ for t and s close to t_* (see Assumption A2).

Let us now explain the second difference. In [3] the authors assumed that (2.2) is satisfied with Ξ and $A(t_0)$ such that $\boldsymbol{w}^\top \Xi\, A^\top(t_0)\, \boldsymbol{w} > 0$, where $\boldsymbol{w}$ is some specific vector. As it turns out, this assumption is rather strong. To lift it, one may consider extending the expansion (2.2) to the second order, namely

$$A(t) = A(t_0) - |t - t_0|^{\beta'}\, \Xi - |t - t_0|^{\beta}\, \Theta + o\left(|t - t_0|^{\beta}\right), \tag{2.3}$$

where $\boldsymbol{w}^\top \Xi\, A^\top(t_0)\, \boldsymbol{w} = 0$, but $\boldsymbol{w}^\top \Theta\, \boldsymbol{w} > 0$. The precise conditions under which this extension is possible are presented by Assumptions A2.3 to A2.6.

In regard to the techniques used to work with the vector-valued setting of this contribution, we refer our reader to the introduction of [3] where the authors describe in detail in what aspects and why this case is much different from the one-dimensional. Here we mention in passing that some of the tools crucial for the one-dimensional case are not available in the multivariate setup (such as the Slepian lemma), while others (such as the Borell-TIS & Piterbarg inequalities) have been successfully extended to this case.

Brief organization of the paper. Main results are presented in Section 2.2 with proofs relegated to Section 2.5. The asymptotics of the double crossing probabilities are presented in Section 2.3 with proofs relegated to Appendix. Section 2.4 contains several auxiliary results, most of which are taken from [3] and reproduced here in an adapted form and without proofs for the reader's convenience. We conclude this section by introducing some notation.

Subscripts. Throughout the rest of the paper, the subscript u on any scalar-, vector or matrix-valued function f defined on $\mathbb{R}^n$ or $\mathbb{R}^n \times \mathbb{R}^n$ means, unless specified otherwise, its rescaling by a factor of $u^{-2/\boldsymbol{\nu}}$, that is, $f_u(\boldsymbol{t}) = f(u^{-2/\boldsymbol{\nu}}\boldsymbol{t})$ or $f_u(\boldsymbol{t}, \boldsymbol{s}) = f(u^{-2/\boldsymbol{\nu}}\boldsymbol{t}, u^{-2/\boldsymbol{\nu}}\boldsymbol{s})$, where $\boldsymbol{\nu}$ is defined in (2.7).

Vectors. All vectors (and only them) are written in bold letters, for instance $\boldsymbol{b} = (b_1, \dots, b_d)^\top$, $\boldsymbol{1} = (1, \dots, 1)^\top$ and $\boldsymbol{0} = (0, \dots, 0)^\top$. If $\mathcal{I} \subset \{1, \dots, n\}$ and $\boldsymbol{b} \in \mathbb{R}^n$, by $\boldsymbol{b}_{\mathcal{I}}$ we mean $(b_i)_{i \in \mathcal{I}} \in \mathbb{R}^{|\mathcal{I}|}$ or, by notation abuse, its extension to $\mathbb{R}^n$ by zeroes: $b_i = 0$ for $i \in \mathcal{I}^c$. Unless specified otherwise, all operations on vectors are performed componentwise. For example, $\boldsymbol{a}\,\boldsymbol{b}$ with $\boldsymbol{a}$, $\boldsymbol{b} \in \mathbb{R}^n$ denotes componentwise producs: $(\boldsymbol{ab})_i = a_i b_i$. Similarly for $\boldsymbol{a}/\boldsymbol{b}$, $e^{\boldsymbol{a}}$, or $\lfloor \boldsymbol{a} \rfloor$, denoting a_i/b_i, e^{a_i} and $\lfloor a_i \rfloor$ correspondingly. We write $\boldsymbol{a} \ge \boldsymbol{b}$ if $a_i \ge b_i$ for all $i \in \{1, \dots, n\}$.

Matrices. If $A = (A_{ij})_{1 \le i, j \le d}$ is a $d \times d$ matrix, we shall write A_{IJ} for the submatrix $(A_{ij})_{i \in I, j \in J}$. If $I = J$, we shall occasionally write A_I instead of A_{II}. $\|A\|$ denotes any fixed norm in the space of $d \times d$ matrices. Our formulae shall not depend on the choice of the norm. For $\boldsymbol{w} \in \mathbb{R}^d$, $\operatorname{diag}(\boldsymbol{w})$ stands for the diagonal matrix with entries $w_1, \dots, w_d$ on the main diagonal.

Asymptotic equivalence. If (X, d_X) is a metric space, $(N, \|\cdot\|_N)$ and $(H, \|\cdot\|_H)$ are normed spaces, $f, g\colon M \to N$ and $h\colon M \to H$, we write "$f = g + o(h)$ as $x \to x_0$" if for every $\varepsilon > 0$ there exists some $\delta > 0$ such that

$$d_X(x, x_0) < \delta \implies \|f(x) - g(x)\|_N \le \varepsilon\, \|h(x)\|_H\,.$$

In particular, this convention will be frequently used with $X = \mathbb{R}^n$, $N = \mathbb{R}^{d\times d}$ the space of matrices with Frobenius norm, $H = \mathbb{R}$ with standard distance $|\cdot|$ or $H = \mathbb{R}^{d\times d}$, again with Frobenius norm.

Quadratic programming problem. Let Σ be a $d \times d$ real matrix with inverse Σ^{-1}. If $\boldsymbol{b} \in \mathbb{R}^d \setminus (-\infty, 0]^d$, then by Lemma 2.5 the quadratic programming problem $\Pi_\Sigma(\boldsymbol{b})$

$$\Pi_\Sigma(\boldsymbol{b})\colon \text{minimize } \boldsymbol{x}^\top \Sigma^{-1} \boldsymbol{x} \text{ under the linear constraint } \boldsymbol{x} \geq \boldsymbol{b}$$

has a unique solution $\widetilde{\boldsymbol{b}} \geq \boldsymbol{b}$ and there exists a unique non-empty index set $I \subset \{1, \dots, d\}$ such that

$$\widetilde{\boldsymbol{b}}_I = \boldsymbol{b}_I, \qquad \widetilde{\boldsymbol{b}}_J = \Sigma_{IJ}(\Sigma_{II})^{-1}\,\boldsymbol{b}_I \geq \boldsymbol{b}_J, \qquad \boldsymbol{w}_I = (\Sigma_{II})^{-1}\,\boldsymbol{b}_I > \boldsymbol{0}_I, \qquad \boldsymbol{w}_J = \boldsymbol{0}_J,$$

where $\boldsymbol{w} = \Sigma^{-1}\widetilde{\boldsymbol{b}}$, where coordinates $J = \{1, \dots, d\} \setminus I$ are responsible for the dimension-reduction phenomena, while coordinates belonging to I play an essential role in the exact asymptotics.

Other notation. We use lower case constants c_1, c_2, ... to denote generic constants used in the proofs, whose exact values are not important and can be changed from line to line. The labeling of the constants starts anew in every proof. Similarly, ϵ_1, ϵ_2, ... denote error terms, that is, functions of various variables which are small in some specific sense, always described near the point where they are introduced. Their labeling also starts anew in every proof.

2.2 Main results

Let $\boldsymbol{X}(\boldsymbol{t})$, $\boldsymbol{t} \in [\boldsymbol{0}, \boldsymbol{T}]$, $\boldsymbol{T} > \boldsymbol{0}$ be a non-stationary centered Gaussian random field with continuous sample paths. Define two matrix-valued functions by

$$R(\boldsymbol{t}, \boldsymbol{s}) = \mathbb{E}\left\{\boldsymbol{X}(\boldsymbol{t})\,\boldsymbol{X}(\boldsymbol{s})^\top\right\}, \qquad \Sigma(\boldsymbol{t}) = R(\boldsymbol{t}, \boldsymbol{t})$$

and assume that $\Sigma(\boldsymbol{t})$ is non-singular. Set $\Sigma = \Sigma(\boldsymbol{0})$. It is known that the function $\sigma_{\boldsymbol{b}}^2(\boldsymbol{t})$, defined by

$$\sigma_{\boldsymbol{b}}^{-2}(\boldsymbol{t}) = \min_{\boldsymbol{x} \geq \boldsymbol{b}} \boldsymbol{x}^\top\, \Sigma^{-1}(\boldsymbol{t})\, \boldsymbol{x} \tag{2.4}$$

and further refered to as the generalized variance, plays a similar role in the multivariate setup to that of the usual variance in the one-dimensional case. Recall that $\boldsymbol{b} \in \mathbb{R}^d$ here is a constant vector with at least one positive component. More precisely, the high exceedance event usually happens near the maximizer of $\sigma_{\boldsymbol{b}}^{-2}(\boldsymbol{t})$. The asymptotics then is determined by the behaviour of $R(\boldsymbol{t}, \boldsymbol{s})$ and $\Sigma(\boldsymbol{t})$ near the this maximiser. Let $\boldsymbol{b}(\boldsymbol{t})$ denote the vector which minimizes (2.4). We shall assume that:

A1 $\sigma_{\boldsymbol{b}}^2(\boldsymbol{t})$ attains its unique maximum at $\boldsymbol{t}_* = \boldsymbol{0}$.

A2 There exist

1. collections of real $d \times d$ matrices $(A_{k,i})_{i=1,\dots,n}$, $k = 1, \dots, 5$ and $(A_{6,i,j})_{i,j=1,\dots,n}$
2. vectors $\boldsymbol{\beta}'$, $\boldsymbol{\beta} \in \mathbb{R}_+^n$ satisfying $\boldsymbol{0} < \boldsymbol{\beta}' < \boldsymbol{\beta} \leq 2\boldsymbol{\beta}'$
3. a vector $\boldsymbol{\alpha} \in (0, 2)^n$

such that

$$\Sigma - R(\boldsymbol{t},\boldsymbol{s}) \sim \sum_{i=1}^{n} \Big[A_{1,i}\, t_i^{\beta_i'} + A_{2,i}\, t_i^{\beta_i} + A_{3,i}\, s_i^{\beta_i'} + A_{4,i}\, s_i^{\beta_i} + S_{\alpha_i, A_{5,i}}(t_i - s_i) \Big] + \sum_{i,\,j=1}^{n} A_{6,i,j}\, t_i^{\beta_i'}\, s_j^{\beta_j'}, \quad \text{(A2.1)}$$

where

$$S_{\alpha,V}(t) \coloneqq |t|^{\alpha} \Big(V \mathbb{1}_{t\ge 0} + V^{\top} \mathbb{1}_{t<0} \Big) \quad (2.5)$$

and $\sim$ means that the error ϵ satisfies

$$\epsilon = o\left(\sum_{i=1}^{n} \Big[t_i^{\beta_i} + s_i^{\beta_i} + |t_i - s_i|^{\alpha_i} \Big] \right) \quad \text{as} \quad (\boldsymbol{t},\boldsymbol{s}) \downarrow (\mathbf{0},\mathbf{0}). \quad \text{(A2.2)}$$

Denote

$$\mathcal{F} \coloneqq \big\{ i \in \{1,\dots,n\} \colon 2\beta_i' = \beta_i \big\}, \qquad \mathcal{I} \coloneqq \{ i \in \{1,\dots,n\} \colon \alpha_i < \beta_i \}, \quad (2.6)$$

and assume further that

$$A_{1,i}\, \boldsymbol{w} = \mathbf{0} \quad \text{and} \quad \boldsymbol{b}(\boldsymbol{t}) - \boldsymbol{b}(\mathbf{0}) = o\left(\sum_{i=1}^{n} t_i^{\beta_i'} \right) \qquad \text{as } \boldsymbol{t} \to \mathbf{0}, \quad \text{(A2.3)}$$

$$\xi_i \coloneqq \boldsymbol{w}^{\top} A_{2,i}\, \boldsymbol{w} > 0 \qquad \text{for all } i \in \{1,\dots,n\}, \quad \text{(A2.4)}$$

$$\varkappa_i \coloneqq \boldsymbol{w}^{\top} A_{5,i}\, \boldsymbol{w} > 0 \qquad \text{for all } i \in \mathcal{I}. \quad \text{(A2.5)}$$

Finally, define a block matrix $D = (D_{i,j})_{i,j\in\mathcal{F}}$, each block of which is a $d \times d$ matrix given by

$$D_{i,j} \coloneqq A_{6,i,j} + A_{1,i}\, \Sigma^{-1} A_{1,j}^{\top}, \quad i,\, j \in \mathcal{F}$$

and assume that it is positive definite, abbreviated below as

$$D \succcurlyeq 0. \quad \text{(A2.6)}$$

A3 There exist $\boldsymbol{\gamma} \in (0,2]^n$ and $C > 0$, such that for all $\boldsymbol{t},\boldsymbol{s}$

$$\mathbb{E}\Big\{ |\boldsymbol{X}(\boldsymbol{t}) - \boldsymbol{X}(\boldsymbol{s})|^2 \Big\} \le C \sum_{i=1}^{n} |t_i - s_i|^{\gamma_i}. \quad \text{(A3)}$$

We shall also frequently use the following notation:

$$\boldsymbol{\nu} \coloneqq \min\{\boldsymbol{\alpha},\boldsymbol{\beta}\}, \quad \mathcal{J} \coloneqq \{ i \in \{1,\dots,n\} \colon \alpha_i = \beta_i \}, \quad \mathcal{K} \coloneqq \{ i \in \{1,\dots,n\} \colon \alpha_i > \beta_i \}. \quad (2.7)$$

Note that

$$\{ i \colon \nu_i = \alpha_i \} = \mathcal{I} \cup \mathcal{J} \quad \text{and} \quad \{ i \colon \nu_i = \beta_i \} = \mathcal{J} \cup \mathcal{K}.$$

Remark 2.1. *It follows from A2.1 that $A_{3,i} = A_{1,i}^{\top}$, $A_{4,i} = A_{2,i}^{\top}$ and $A_{6,i,j}^{\top} = A_{6,j,i}$. Moreover, the terms with $t_i^{\beta_i'}\, t_j^{\beta_j'}$ such that $i \notin \mathcal{F}$, $j \notin \mathcal{F}$ or both can be subsumed into the error term. Hence, the assumption A2.1 may be rewritten as follows:*

$$\Sigma - R(\boldsymbol{t},\boldsymbol{s}) = \sum_{i=1}^{n} \Big[A_{1,i}\, t_i^{\beta_i'} + A_{2,i}\, t_i^{\beta_i} + A_{1,i}^{\top}\, s_i^{\beta_i'} + A_{2,i}^{\top}\, s_i^{\beta_i} + S_{\alpha_i, A_{5,i}}(t_i - s_i) \Big] + \sum_{i,\,j\in\mathcal{F}} A_{6,i,j}\, t_i^{\beta_i/2}\, s_j^{\beta_j/2} + o\left(\sum_{i=1}^{n} \Big[t_i^{\beta_i} + s_i^{\beta_i} + |t_i - s_i|^{\alpha_i} \Big] \right). \quad (2.8)$$

Remark 2.2. *It may be instructive to compare our assumption A2 to that of [3]. To this end, consider the case* $A_{1,i} = A_{6,i,j} = 0$*. The assumptions A2.3 and A2.6 are fulfilled automatically and assumption A2.1 reads*

$$\Sigma - R(\boldsymbol{t}, \boldsymbol{s}) = \sum_{i=1}^{n} \Big[A_{2,i}\, t_i^{\beta_i} + A_{2,i}^\top\, s_i^{\beta_i} + S_{\alpha_i, A_{5,i}}(t_i - s_i) \Big] + \epsilon, \tag{A2.1*}$$

with the same error order as in A2.2. Assumption A2.1 combines (2.10), (2.11) and (2.13) of the aforementioned paper into one and extends it from processes to fields of simple (additive) covariance structure. It also has an advantage over them since it does not require finding* $A(\boldsymbol{t})$ *such that* $\Sigma(\boldsymbol{t}) = A(\boldsymbol{t})\, A^\top(\boldsymbol{t})$*, which can always be done in theory, but hard to implement in practice. Next, assumption A2.4 extends (2.12) to the case of fields and A2.5 does the same to the condition* $\boldsymbol{w}^\top V \boldsymbol{w} > 0$ *of Theorem 2.4.*

Another important difference consists in allowing the leading order of $\Sigma - R(\boldsymbol{t}, \boldsymbol{s})$ *to nullify* $\boldsymbol{w}$*, that is, to fail the condition* $\boldsymbol{w}^\top A_{1,i}\, \boldsymbol{w} > 0$ *and even* $A_{1,i}\, \boldsymbol{w} \neq \boldsymbol{0}$[1]*. In this case we require a certain speed of convergence of the quadratic programming problem solutions* $\boldsymbol{b}(\boldsymbol{t})$ *to* $\boldsymbol{b}(\boldsymbol{0})$*, see A2.3.*

Remark 2.3. *Assumption A2.6 is somewhat mysterious, which is why we present a few intermediary results without assuming it in the Appendix. By Lemma 2.16 it is equivalent to the following: there exists a family* $(C_{i,k})_{i,k \in \mathcal{F}}$ *such that*

$$D_{i,j} = \sum_{k \in \mathcal{F}} C_{i,k}\, C_{k,j}^\top. \tag{2.9}$$

Note that if $A_{6,i,j} = 0$*, then* $A_{1,i}\Sigma^{-1} A_{1,j}^\top = D_i\, D_j^\top$ *with* $D_i = A_{1,i}\Sigma^{-1/2}$*, and therefore the assumption is satisfied. An example of this situation may be found in our fBm double crossing example, see Section 2.3.2. Another useful example is when* $A_{6,i,j}$ *it not zero, but can itself be represented as* $C_i\, C_j^\top$ *for some* C_i*.* $A_{6,i,j} = C_i\, C_j^\top$*, then the assumption is also satisfied.*

2.2.1 Constants

For $\alpha \in (0, 2]$ and a matrix V, satisfying standard assumptions, let $\boldsymbol{Y}_{\alpha,V}(t)$, $\boldsymbol{t} \in \mathbb{R}$ be a multivariate fBm with cmf

$$R_{\alpha,V}(t, s) \coloneqq S_{\alpha,V}(t) + S_{\alpha,V}(-s) - S_{\alpha,V}(t - s), \tag{2.10}$$

where $S_{\alpha,V}$ is defined by (2.5).

For a triplet of disjoint sets $\mathcal{I},\ \mathcal{J},\ \mathcal{K} \subset \{1, \dots, n\}$, vector $\boldsymbol{\nu} \in (0, 2]^{\mathcal{I} \cup \mathcal{J} \cup \mathcal{K}}$ and two collections of matrices $\mathbb{V} = (V_i)_{i \in \mathcal{I} \cup \mathcal{J}}$ and $\mathbb{W} = (W_i)_{i \in \mathcal{J} \cup \mathcal{K}}$ define a multivariate additive fBm field $\boldsymbol{Y}_{\boldsymbol{\nu},\mathbb{V}}(\boldsymbol{t})$, $\boldsymbol{t} \in \mathbb{R}^n$ and a deterministic vector field $\boldsymbol{d}_{\boldsymbol{\nu},\mathbb{W}}(\boldsymbol{t})$, $\boldsymbol{t} \in \mathbb{R}^n$ by

$$\boldsymbol{Y}_{\boldsymbol{\nu},\mathbb{V}}(\boldsymbol{t}) \coloneqq \sum_{i \in \mathcal{I} \cup \mathcal{J}} \Big[\boldsymbol{Y}_{\nu_i, V_i}(t_i) - S_{\nu_i, V_i}(t_i)\, \boldsymbol{1} \Big], \qquad \boldsymbol{d}_{\boldsymbol{\nu},\mathbb{W}}(\boldsymbol{t}) \coloneqq \sum_{i \in \mathcal{J} \cup \mathcal{K}} |t_i|^{\nu_i}\, W_i\, \boldsymbol{1}.$$

Consider also a family of matrices $\mathbb{D} = (C_{i,j})_{i,j \in \mathcal{D}}$, $\mathcal{D} \subset \{1, \dots, n\}$, and set

$$C_k(\boldsymbol{t}) \coloneqq \sum_{i \in \mathcal{D}} C_{i,k}\, t_i^{\beta_i/2}, \quad \boldsymbol{Z}(\boldsymbol{t}) \coloneqq \sum_{k \in \mathcal{D}} C_k(\boldsymbol{t})\, \boldsymbol{\mathcal{N}}_k,$$

where $\boldsymbol{\mathcal{N}}_k \sim N(\boldsymbol{0}, I)$ are standard Gaussian vectors, independent of each other and of the fields $\boldsymbol{Y}_{\nu_i, V_i}$, $i \in \mathcal{I} \cup \mathcal{J}$.

[1] Note that $A = 0 \implies A\, \boldsymbol{w} = \boldsymbol{0} \implies \boldsymbol{w}^\top A\, \boldsymbol{w} = 0$, but neither implication is reversible.

Finally, for a compact set $E \subset \mathbb{R}^n$ define

$$H_{\boldsymbol{\nu},\mathbb{V},\mathbb{W},\mathbb{D}}\,(E) := \int_{\mathbb{R}^d} e^{\mathbf{1}^\top \boldsymbol{x}}\, \mathbb{P}\left\{\exists\, \boldsymbol{t} \in E \colon \mathbf{Y}_{\boldsymbol{\nu},\mathbb{V}}(\boldsymbol{t}) + \boldsymbol{Z}(\boldsymbol{t}) - \boldsymbol{d}_{\boldsymbol{\nu},\mathbb{W}}(\boldsymbol{t}) > \boldsymbol{x}\right\} d\boldsymbol{x}$$

and

$$\mathcal{H}_{\boldsymbol{\nu},\mathbb{V},\mathbb{W},\mathbb{D}} := \lim_{\Lambda\to\infty} \lim_{S\to\infty} S^{-|\mathcal{I}|} H_{\boldsymbol{\nu},\mathbb{V},\mathbb{W},\mathbb{D}}\left([\mathbf{0}, \boldsymbol{S}']\right), \qquad \boldsymbol{S}' = S\mathbf{1}_{\mathcal{I}} + \Lambda \mathbf{1}_{\mathcal{I}^c}$$

whenever the limit exists. Define also

$$H_{\boldsymbol{\nu},\mathbb{V},\mathbb{W}}(E) := H_{\boldsymbol{\nu},\mathbb{V},\mathbb{W},\varnothing}(E),$$

and similarly for $\mathcal{H}_{\boldsymbol{\nu},\mathbb{V},\mathbb{W}}$.

As we shall see below, for $\mathcal{I}$, $\mathcal{J}$, $\mathcal{K}$, $\mathcal{F}$ from (2.6) and (2.7) and matrices from A2.1 this limit exists and is positive and finite, provided that A2.4 and A2.5 are satisfied, see Theorem 2.1.

For an $n \times n$ matrix Ξ and vector $\boldsymbol{\beta} > \mathbf{0}$, define

$$G(\boldsymbol{\beta}, \Xi) := \int_{\mathbb{R}^n_+} \exp\left(-\frac{1}{2}\sum_{i,j} \Xi_{i,j}\, t_i^{\beta_i/2}\, t_j^{\beta_j/2}\right) d\boldsymbol{t}\,. \tag{2.11}$$

2.2.2 Main theorem

Theorem 2.1. *Let $\boldsymbol{X}(\boldsymbol{t})$, $\boldsymbol{t} \in [\mathbf{0}, \boldsymbol{T}] \subset \mathbb{R}^n$ be a centered $\mathbb{R}^d$-valued Gaussian random field, satisfying Assumptions A1, A2 and A3. Then*

$$\mathbb{P}\left\{\exists\, \boldsymbol{t} \in [\mathbf{0}, \boldsymbol{T}] \colon \boldsymbol{X}(\boldsymbol{t}) > u\boldsymbol{b}\right\} \sim \mathcal{H}_{\boldsymbol{\nu},\mathbb{V}_{\boldsymbol{w}},\mathbb{W}_{\boldsymbol{w}},\mathbb{D}_{\boldsymbol{w}}}\, G(\boldsymbol{\beta}_{\mathcal{I}}, \Xi_{\mathcal{I},\mathcal{I}})\, u^{\zeta}\, \mathbb{P}\left\{\boldsymbol{X}(\mathbf{0}) > u\boldsymbol{b}\right\}, \tag{2.12}$$

with

$$\boldsymbol{\nu} = \min\{\boldsymbol{\alpha}, \boldsymbol{\beta}\}, \quad \zeta = \sum_{i\in\mathcal{I}} \left(\frac{2}{\alpha_i} - \frac{2}{\beta_i}\right),$$

$$\mathbb{V}_{\boldsymbol{w}} = \Big(\operatorname{diag}(\boldsymbol{w})\, A_{5,i}\, \operatorname{diag}(\boldsymbol{w})\Big)_{i\in\mathcal{I}\cup\mathcal{J}}, \quad \mathbb{W}_{\boldsymbol{w}} = \Big(\operatorname{diag}(\boldsymbol{w})\, A_{2,i}\, \operatorname{diag}(\boldsymbol{w})\Big)_{i\in\mathcal{J}\cup\mathcal{K}},$$

$$\mathbb{D}_{\boldsymbol{w}} = \Big(\operatorname{diag}(\boldsymbol{w})\, C_{i,k}\Big)_{i,k\in(\mathcal{J}\cup\mathcal{K})\cap\mathcal{F}}, \qquad \Xi_{i,j} = \boldsymbol{w}^\top \Big[2\, A_{2,i}\, \mathbb{1}_{i=j} + D_{i,j}\, \mathbb{1}_{i,j\in\mathcal{F}}\Big]\boldsymbol{w},$$

G defined by (2.11)*, $(C_{i,k})_{i,k\in(\mathcal{J}\cup\mathcal{K})\cap\mathcal{F}}$ any family of matrices satisfying* (2.9)*, and*

$$G(\boldsymbol{\beta}_{\mathcal{I}}, \Xi_{\mathcal{I},\mathcal{I}}), \quad \mathcal{H}_{\boldsymbol{\nu},\mathbb{V}_{\boldsymbol{w}},\mathbb{W}_{\boldsymbol{w}},\mathbb{D}_{\boldsymbol{w}}} \in (0, \infty).$$

Remark 2.4. *By Lemma* (2.18)*, the matrix Ξ may be alternatively defined by*

$$\sigma_{\boldsymbol{b}}^{-2}(\boldsymbol{\tau}) - \sigma_{\boldsymbol{b}}^{-2}(\mathbf{0}) = \sum_{i,j=1}^{n} \Xi_{i,j}\, \tau_i^{\beta_i/2}\, \tau_j^{\beta_j/2} + o\left(\sum_{i=1}^{n} \tau_i^{\beta_i}\right).$$

This is useful from the practical point of view, since to apply the theorem we first have to compute $\sigma_{\boldsymbol{b}}^{-2}(\boldsymbol{\tau})$.

Corollary 2.1. *If the conditions of Theorem 2.1 are satisfied with $\mathcal{F} = \varnothing$ or $D_{i,j} = 0$ for all $i, j \in \mathcal{F}$, then*

$$\mathbb{P}\left\{\exists\, \boldsymbol{t} \in [\mathbf{0}, \boldsymbol{T}] \colon \boldsymbol{X}(\boldsymbol{t}) > u\boldsymbol{b}\right\} \sim \mathcal{H}_{\boldsymbol{\nu},\mathbb{V}_{\boldsymbol{w}},\mathbb{W}_{\boldsymbol{w}}} \prod_{i\in\mathcal{I}} (2\,\xi_i)^{-1/\beta_i}\, \Gamma\left(\frac{1}{\beta_i} + 1\right) u^{\zeta}\, \mathbb{P}\{\boldsymbol{X}(\mathbf{0}) > u\boldsymbol{b}\}.$$

If $\mathcal{F} \subset \mathcal{I}$, but not necessarily empty, then $\mathcal{D} = \varnothing$ and therefore (2.12) *holds with $\mathcal{H}_{\boldsymbol{\nu},\mathbb{V}_{\boldsymbol{w}},\mathbb{W}_{\boldsymbol{w}}}$.*

Remark 2.5. *If we consider the same problem on $[-\boldsymbol{T}_1, \boldsymbol{T}_2]$, where $\boldsymbol{T}_{1,2} \geq \boldsymbol{0}$ satisfy $T_{1,i}+T_{2,i} > 0$ for all i (so that the rectangle $[-\boldsymbol{T}_1, \boldsymbol{T}_2]$ be of full dimension), we can obtain a result similar to Theorem 2.1 under slightly modified assumptions. Denote*

$$\mathcal{L} \coloneqq \{i \colon T_{1,i} > 0\}, \qquad \mathcal{R} \coloneqq \{i \colon T_{2,i} > 0\},$$

and consider the following symmetric extension of A2.1 to the negative values of t_i's:

$$\Sigma - R(\boldsymbol{t}, \boldsymbol{s}) \sim \sum_{i=1}^{n} \Big[A_{1,i} \, |t_i|^{\beta_i'} + A_{2,i} \, |t_i|^{\beta_i} + A_{3,i} \, |s_i|^{\beta_i'} + A_{4,i} \, |s_i|^{\beta_i} + S_{\alpha_i, A_{5,i}}(t_i - s_i) \Big] + \sum_{i,\, j=1}^{n} A_{6,i,j} \, |t_i|^{\beta_i'} \, |s_j|^{\beta_j'}.$$

With these assumptions we have

$$\mathbb{P}\left\{ \exists\, \boldsymbol{t} \in [-\boldsymbol{T}_1, \boldsymbol{T}_2] \colon \boldsymbol{X}(\boldsymbol{t}) > u\boldsymbol{b} \right\} \sim u^{\zeta} \, \mathcal{H}_{\boldsymbol{\nu}, \mathbb{V}_{\boldsymbol{w}}, \mathbb{W}_{\boldsymbol{w}}, \mathbb{D}_{\boldsymbol{w}}}(\mathcal{L}, \mathcal{R}) \, G(\boldsymbol{\beta}_{\mathcal{I}}, \Xi_{\mathcal{I},\mathcal{I}}) \, \mathbb{P}\left\{ \boldsymbol{X}(\boldsymbol{0}) > u\boldsymbol{b} \right\},$$

where

$$\mathcal{H}_{\boldsymbol{\nu}, \mathbb{V}, \mathbb{W}, \mathbb{D}}(\mathcal{L}, \mathcal{R}) \coloneqq \lim_{\Lambda \to \infty} \lim_{S \to \infty} S^{-|\mathcal{I}|} H_{\boldsymbol{\nu}, \mathbb{V}, \mathbb{W}, \mathbb{D}}\left(\boldsymbol{S}' \left[-\boldsymbol{1}_{\mathcal{L}}, \boldsymbol{1}_{\mathcal{R}} \right] \right) \in (0, \infty), \qquad \boldsymbol{S}' \coloneqq S\boldsymbol{1}_{\mathcal{I}} + \Lambda \boldsymbol{1}_{\mathcal{I}^c}.$$

2.3 Double crossing probabilities

Let $X(t)$, $t \in [0, T] \subset \mathbb{R}$ be a continuous centered Gaussian process, with covariance function $r(t, s) = \mathbb{E}\left\{ X(t) X(s) \right\}$. We want to study the probability that in a given finite time interval the process X hits two distant barriers: one above and one below its initial point. Formally, we study the asymptotics as $u \to \infty$ of

$$\mathbb{P}\left\{ \exists\, t,\, s \in [0, T] \colon X(t) > au,\ X(s) < -bu \right\}, \qquad a, b > 0,$$

which we shall refer to as the double crossing probability. For our purposes, it may be conveniently rewritten as

$$\mathbb{P}\left\{ \exists\, \boldsymbol{t} \in [0, T]^2 \colon \boldsymbol{X}(\boldsymbol{t}) > u\boldsymbol{b} \right\}, \quad \boldsymbol{X}(\boldsymbol{t}) = \left(X(t_1),\ -X(t_2) \right)^{\top}, \quad \boldsymbol{b} = (a, b)^{\top}.$$

The problem is thus reduced to the study of a simple (not double) crossing of a two-dimensional vector-field $\boldsymbol{X}(\boldsymbol{t})$ over $[0, T]^2$, which is exactly the setup of our Main Theorem 2.1.

Let us briefly show how Main Theorem 2.1 may be applied in this case. First, we should find the maximizer of the generalized variance $\sigma^2_{a,b}(\boldsymbol{t})$ defined by

$$\sigma^{-2}_{a,b}(\boldsymbol{t}) = \min_{\boldsymbol{x} \geq (a,b)^{\top}} \boldsymbol{x}^{\top} \, \Sigma^{-1}(\boldsymbol{t}) \, \boldsymbol{x}, \tag{2.13}$$

where the matrix $\Sigma(\boldsymbol{t})$ is given by $\Sigma(\boldsymbol{t}) = R(\boldsymbol{t}, \boldsymbol{t})$ and

$$R(\boldsymbol{t}, \boldsymbol{s}) = \mathbb{E}\left\{ \boldsymbol{X}(\boldsymbol{t}) \, \boldsymbol{X}(\boldsymbol{s})^{\top} \right\} = \begin{pmatrix} r(t_1, s_1) & -r(t_1, s_2) \\ -r(t_2, s_1) & r(t_2, s_2) \end{pmatrix}, \qquad \boldsymbol{t} = (t_1, t_2)^{\top}, \quad \boldsymbol{s} = (s_1, s_2)^{\top}.$$

The inverse of $\Sigma(\boldsymbol{t})$ exists for all $\boldsymbol{t} \not\in \{\boldsymbol{t} = (t, t)^{\top} : t \in [0, T]\}$ and is given by

$$\Sigma^{-1}(\boldsymbol{t}) = \frac{1}{r(t_1, t_1) \, r(t_2, t_2) - r^2(t_1, t_2)} \begin{pmatrix} r(t_2, t_2) & r(t_1, t_2) \\ r(t_1, t_2) & r(t_1, t_1) \end{pmatrix}.$$

Near the diagonal $\{\boldsymbol{t} = (t,t)^\top : t \in [0,T]\}$ the matrix $\Sigma(\boldsymbol{t})$ is degenerate, so we need some additional lemma to deal with the probability that the extreme event happens there. Intuitively, such event means that a process has managed to hit both boundaries during a very short time interval, which seems unlikely. The precise meaning to this is given by the next lemma.

Lemma 2.1. *Let $X(t)$, $t \in [0,T]$ be a centered Gaussian process with a.s. continuous sample paths. If there exists a function f such that*

$$\mathbb{E}\left\{\Big[X(t+l) - X(t)\Big]^2\right\} < f(l)$$

for all $t \in [0,T]$, $l > 0$ and $f(l) \to 0$ as $l \to 0$, then for any $\delta > 0$ there exists $\varepsilon > 0$ such that

$$\mathbb{P}\left\{\exists\, t_1,\, t_2 \in [0,T],\ |t_1 - t_2| < \varepsilon \colon X(t_1) > au,\ X(t_2) < -bu\right\} = o\left(e^{-\delta u^2}\right).$$

Unfortunately, even the problem of minimizing $\sigma_{\boldsymbol{b}}^{-2}(\boldsymbol{t})$ over $\{\boldsymbol{t} = (t,s)^\top : |t-s| > \varepsilon\}$ is too hard in its full generality. In the next two sections we study the double crossing probabilities for two classes of processes: stationary with positive correlation and fractional Brownian motion.

2.3.1 Double crossing probabilities: stationary case

Let X be a stationary Gaussian process with unit variance and positive correlation function $r(t,s) = \rho(|t-s|) > 0$ satisfying

$$\rho(t) = 1 - \vartheta\, t^\alpha + o\left(t^\alpha\right) \quad \text{as} \quad t \to 0 \tag{2.14}$$

with some $\vartheta > 0$ and $\alpha \in (0,2]$. We additionally assume that ρ is strictly decreasing and differentiable in $t > 0$.

We need to minimize

$$\frac{x_1^2 + 2x_1x_2\,\rho(|t_1-t_2|) + x_2^2}{1 - \rho^2(t_1,t_2)}$$

with respect to $\boldsymbol{x} = (x_1,x_2)^\top$ subject to $\boldsymbol{x} \geq (a,b)^\top$, and then minimize it again, but with respect to $\boldsymbol{t}$ sufficiently far away from the diagonal. The unique solution of the first minimization problem in this case is $\boldsymbol{x} = (a,b)^\top$. Therefore, we have

$$\sigma_{a,b}^{-2}(\boldsymbol{t}) = \min_{\boldsymbol{x} \geq (a,b)^\top} \boldsymbol{x}^\top\, \Sigma^{-1}(\boldsymbol{t})\, \boldsymbol{x} = \frac{a^2 + 2ab\,\rho(|t_1-t_2|) + b^2}{1 - \rho^2(|t_1-t_2|)}.$$

To solve the second, we note by rewriting $\sigma_{a,b}^{-2}(\boldsymbol{t})$ as

$$\sigma_{a,b}^{-2}(\boldsymbol{t}) = \frac{(a+b)^2}{1 - \rho^2(|t_1-t_2|)} - \frac{2ab}{1 + \rho(|t_1-t_2|)}$$

that $\sigma_{a,b}^{-2}(\boldsymbol{t})$ attains its minimum at the same point as $\rho(|t_1-t_2|)$. Since ρ is decreasing, we have two minimizing points $\boldsymbol{t}_{*,1} = (0,T)^\top$ and $\boldsymbol{t}_{*,2} = (T,0)^\top$ and

$$\sigma_{a,b}^{-2}(\boldsymbol{t}_{*,1}) = \sigma_{a,b}^{-2}(\boldsymbol{t}_{*,2}) = \frac{a^2 + 2ab\rho(T) + b^2}{1 - \rho^2(T)}.$$

By Lemma 2.1 and Piterbarg inequality, we can show that for any $\varepsilon > 0$ we have

$$\mathbb{P}\left\{\exists\, \boldsymbol{t} \in [\boldsymbol{0}, \boldsymbol{T}] \colon \boldsymbol{X}(\boldsymbol{t}) > u\, \boldsymbol{b}\right\}$$

$$\sim \mathbb{P}\left\{\exists\, \boldsymbol{t} \in [0,\varepsilon] \times [T-\varepsilon, T] \colon \boldsymbol{X}(\boldsymbol{t}) > u\, \boldsymbol{b}\right\} + \mathbb{P}\left\{\exists\, \boldsymbol{t} \in [T-\varepsilon, T] \times [0,\varepsilon] \colon \boldsymbol{X}(\boldsymbol{t}) > u\, \boldsymbol{b}\right\}$$

$$= \mathbb{P}\left\{\exists\, \boldsymbol{t} \in [0,\varepsilon]^2 \colon \boldsymbol{X}_1(\boldsymbol{t}) > u\, \boldsymbol{b}\right\} + \mathbb{P}\left\{\exists\, \boldsymbol{t} \in [0,\varepsilon]^2 \colon \boldsymbol{X}_2(\boldsymbol{t}) > u\, \boldsymbol{b}\right\},$$

where in the last line we performed appropriate time changes in both probabilities by introducing $\boldsymbol{X}_1(\boldsymbol{t}) = (X(t_1), -X(T-t_2))^\top$ and $\boldsymbol{X}_2(\boldsymbol{t}) = (X(T-t_1), -X(t_2))^\top$.

In order to apply Main Theorem 2.1 to

$$\mathbb{P}\left\{\exists\, \boldsymbol{t} \in [0,\varepsilon]^2 \colon \boldsymbol{X}_1(\boldsymbol{t}) > u\, \boldsymbol{b}\right\} \quad \text{and} \quad \mathbb{P}\left\{\exists\, \boldsymbol{t} \in [0,\varepsilon]^2 \colon \boldsymbol{X}_2(\boldsymbol{t}) > u\, \boldsymbol{b}\right\},$$

we need to derive asymptotic expansions of the corresponding covariances. This is done in the following lemma.

Lemma 2.2. *The random field $\boldsymbol{X}_1$ satisfies the assumptions A1 to A3 of Theorem 2.1 with*

$$\alpha_1 = \alpha_2 = \alpha, \quad \beta_1 = \beta_2 = 1, \quad \mathcal{F} = \varnothing$$

and

$$A_{2,1} = A_{2,2}^\top = -\rho'(T)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A_{5,1} = \vartheta \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_{5,2} = \vartheta \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Moreover,

$$\boldsymbol{w} = \Sigma^{-1}(\boldsymbol{0})\,(a,b)^\top = \frac{1}{1-\rho^2(T)}\begin{pmatrix} 1 & \rho(T) \\ \rho(T) & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{1-\rho^2(T)}\begin{pmatrix} a + b\rho(T) \\ b + a\rho(T) \end{pmatrix}$$

Assumption A2.4 is satisfied with ξ_i given by

$$\xi_1 = \boldsymbol{w}^\top A_{2,1}\, \boldsymbol{w} = \xi_2 = \boldsymbol{w}^\top A_{2,2}\, \boldsymbol{w} = \frac{-\rho'(T)(a+b\rho(T))(b+a\rho(T))}{(1-\rho^2(T))^2} > 0,$$

and A2.5 with $\varkappa_i$ given by

$$\varkappa_1 = \boldsymbol{w}^\top A_{5,1}\, \boldsymbol{w} = \varkappa_2(\boldsymbol{w}) = \boldsymbol{w}^\top A_{5,2}\, \boldsymbol{w} = \frac{C(b+a\rho(T))^2}{(1-\rho^2(T))^2} > 0.$$

Using Lemma 2.1, and noting that $\mathcal{F} = \varnothing$, we may apply the first assertion of Corollary 2.1 instead of Theorem 2.1, and obtain the following theorem on the asymptotics of double crossing probabilities.

Theorem 2.2. *Let $X(t)$, $t \in [0,T]$ be a centered a.s. continuous stationary Gaussian process with unit variance and positive strictly decreasing and differentiable correlation function $\rho(t) > 0$ which satisfies (2.14) with some $\vartheta > 0$ and $\alpha \in (0,2]$. Define*

$$p(u) \coloneqq \mathbb{P}\left\{X(0) > au,\ X(T) < -bu\right\}.$$

Then

Pickands case *If $\alpha < 1$,*

$$\mathbb{P}\left\{\exists\, t,\, s \in [0,T] \colon X(t) > au,\ X(s) < -bu\right\} \sim C\, \mathcal{H}^2\, u^{4/\alpha - 2}\, p(u),$$

where

$$\mathcal{H} = \lim_{S\to\infty} \frac{1}{S}\, \mathbb{E}\left\{\sup_{t\in[0,S]} e^{B_H(t) - t^{2H}/2}\right\}, \quad C = \frac{2^{1+2/\alpha}\, \theta^{2/\alpha}}{(\rho'(T))^2}\left(\frac{(1-\rho^2(T))^2}{(a+b\rho(T))(b+a\rho(T))}\right)^{2-2/\alpha}.$$

Piterbarg case *If $\alpha = 1$,*

$$\mathbb{P}\left\{\exists\, t, s \in [0,T]\colon X(t) > au,\ X(s) < -bu\right\} \sim 2\, \widetilde{\mathcal{H}}_1\, \widetilde{\mathcal{H}}_2\, p(u),$$

where

$$\widetilde{\mathcal{H}}_k = \lim_{S\to\infty} \mathbb{E}\left\{\sup_{t\in[0,S]} e^{B(t)-(1+\lambda_k)t/2}\right\}, \qquad \lambda_1 = \frac{-\rho'(T)\, w_2}{2\, w_1\, \vartheta}, \quad \lambda_2 = \frac{-\rho'(T)\, w_1}{2\, w_2\, \vartheta} > 0.$$

Talagrand case *If $\alpha > 1$,*

$$\mathbb{P}\left\{\exists\, t,\, s \in [0,T]\colon X(t) > au,\ X(s) < -bu\right\} \sim 2\, p(u).$$

2.3.2 Double crossing probabilities: fBm case

In this section we study the double crossing probability (2.1) in case when X is a fractional Brownian motion B_H, that is, a Gaussian process associated to the following covariance function:

$$r(t,s) = \frac{1}{2}\Big(t^{2H} + s^{2H} - |t-s|^{2H}\Big).$$

As explained in Section 2.3, we should first minimize $\sigma_{\boldsymbol{b}}^{-2}(\boldsymbol{t})$, defined in (2.13), in $\boldsymbol{x}$ subject to $\boldsymbol{x} \ge (a,b)^\top$, and then minimize it again, but with respect to $\boldsymbol{t}$ sufficiently far away from the diagonal. Minimization in $\boldsymbol{x}$ yields $\boldsymbol{x} = (a,b)^\top$, and therefore we have

$$\sigma_{a,b}^{-2}(\boldsymbol{t}) = \min_{\boldsymbol{x}\ge(a,b)^\top} \boldsymbol{x}^\top\, \Sigma^{-1}(\boldsymbol{t})\, \boldsymbol{x} = \frac{a^2\, t_1^{2H} + 2ab\, r(t_1,t_2) + b^2\, t_2^{2H}}{(t_1 t_2)^{2H} - r^2(t_1,t_2)}.$$

For the second minimization, we have the following lemma.

Lemma 2.3. *There exists $t_* \in (0,T)$, such that the function $\sigma_{a,b}^{-2}(\boldsymbol{t})$ defined on $\{\boldsymbol{t} \in [0,T]^2\colon |t_1 - t_2| > \varepsilon\}$, $\varepsilon > 0$ attains,*

1. *If $a < b$, its unique minimum at point $\boldsymbol{t}_{*,1} = (T, t_*)^\top$*
2. *If $a > b$, its unique minimum at point $\boldsymbol{t}_{*,2} = (t_*, T)^\top$*
3. *If $a = b$, its minimum at exactly two points $\boldsymbol{t}_{*,1} = (T, t_*)^\top$ and $\boldsymbol{t}_{*,2} = (t_*, T)^\top$*

   Moreover, we have

   $$\sigma_{\boldsymbol{b}}^{-2}(\boldsymbol{t}_{*,1}) - \sigma_{\boldsymbol{b}}^{-2}(\boldsymbol{t}_{*,1} - \boldsymbol{\tau}) \sim -\kappa_1\, \tau_1 - \kappa_2\, \tau_2^2 \quad \text{and} \quad \sigma_{\boldsymbol{b}}^{-2}(\boldsymbol{t}_{*,2}) - \sigma_{\boldsymbol{b}}^{-2}(\boldsymbol{t}_{*,2} - \boldsymbol{\tau}) \sim -\kappa_2\, \tau_1^2 - \kappa_1\, \tau_2$$

   with

$$\kappa_1 := -\frac{\partial \sigma_{\boldsymbol{b}}^{-2}}{\partial t_1}(\boldsymbol{t}_{*,1}) = -\frac{\partial \sigma_{\boldsymbol{b}}^{-2}}{\partial t_2}(\boldsymbol{t}_{*,2}) > 0, \qquad \kappa_2 := \frac{\partial^2 \sigma_{\boldsymbol{b}}^{-2}}{\partial t_2^2}(\boldsymbol{t}_{*,1}) = \frac{\partial^2 \sigma_{\boldsymbol{b}}^{-2}}{\partial t_1^2}(\boldsymbol{t}_{*,2}) > 0.$$

By Lemma 2.1 and Piterbarg inequality (2.25), we can show that for any $\varepsilon > 0$ we have

$$\begin{aligned} &\mathbb{P}\left\{\exists\, \boldsymbol{t} \in [\boldsymbol{0}, \boldsymbol{T}]\colon \boldsymbol{X}(\boldsymbol{t}) > u\, \boldsymbol{b}\right\} \\ &\quad \sim \mathbb{P}\left\{\exists\, \boldsymbol{t} \in [T-\varepsilon, T] \times [t_* - \varepsilon, t_* + \varepsilon]\colon \boldsymbol{X}(\boldsymbol{t}) > u\, \boldsymbol{b}\right\} \\ &\qquad + \mathbb{P}\left\{\exists\, \boldsymbol{t} \in [t_* - \varepsilon, t_* + \varepsilon] \times [T-\varepsilon, T]\colon \boldsymbol{X}(\boldsymbol{t}) > u\, \boldsymbol{b}\right\} \qquad (2.15) \\ &\quad = \mathbb{P}\left\{\exists\, \boldsymbol{t} \in [0,\varepsilon] \times [-\varepsilon, \varepsilon]\colon \boldsymbol{X}_1(\boldsymbol{t}) > u\, \boldsymbol{b}\right\} \\ &\qquad + \mathbb{P}\left\{\exists\, \boldsymbol{t} \in [-\varepsilon, \varepsilon] \times [0,\varepsilon]\colon \boldsymbol{X}_2(\boldsymbol{t}) > u\, \boldsymbol{b}\right\}, \end{aligned}$$

where in the last line we performed appropriate time changes in both probabilities by introducing $\boldsymbol{X}_1(\boldsymbol{t}) = (X(T-t_1), -X(t_2-t_*))^\top$ and $\boldsymbol{X}_2(\boldsymbol{t}) = (X(t_1-t_*), -X(T-t_2))^\top$.

Next, assume that $a > b$. Then we can use the Piterbarg inequality (2.25) again to show that

$$\mathbb{P}\left\{\exists\, \boldsymbol{t} \in [-\varepsilon,\varepsilon]\times[0,\varepsilon] \colon \boldsymbol{X}_2(\boldsymbol{t}) > u\,\boldsymbol{b}\right\} = o\big(\mathbb{P}\left\{\exists\, \boldsymbol{t} \in [0,\varepsilon]\times[-\varepsilon,\varepsilon] \colon \boldsymbol{X}_1(\boldsymbol{t}) > u\,\boldsymbol{b}\right\}\big). \tag{2.16}$$

Intuitively this means that the process is less likely to first hit a higher barrier and then hit the lower than the other way around. If $a = b$, the two probabilities are equal, which is clear from the symmetry. Combining (2.15) and (2.16), we obtain that if $a \ge b$, then

$$\mathbb{P}\left\{\exists\, \boldsymbol{t} \in [\boldsymbol{0},\boldsymbol{T}] \colon \boldsymbol{X}(\boldsymbol{t}) > u\boldsymbol{b}\right\} \sim (1+\mathbb{1}_{a=b})\,\mathbb{P}\left\{\exists\, \boldsymbol{t} \in [0,\varepsilon]\times[-\varepsilon,\varepsilon] \colon \boldsymbol{X}_1(\boldsymbol{t}) > u\boldsymbol{b}\right\}.$$

In order to apply Theorem 2.1 to $\mathbb{P}\left\{\exists\, \boldsymbol{t} \in [0,\varepsilon]^2 \colon \boldsymbol{X}_1(\boldsymbol{t}) > u\,\boldsymbol{b}\right\}$, we need to derive asymptotic expansions of the corresponding covariance matrix $R(\boldsymbol{t},\boldsymbol{s})$ as $\boldsymbol{t}$ and $\boldsymbol{s}$ tend to zero. This is done in the following lemma.

Lemma 2.4. *The random field $\boldsymbol{X}_1(\boldsymbol{t})$ satisfies the assumptions A1 to A3 of Theorem 2.1 with*

$$\alpha_1 = \alpha_2 = 2H, \quad \beta_1 = 1, \quad \beta_2 = 2, \quad \beta_2' = 1, \quad \mathcal{F} = \{2\},$$

and

$$A_{2,1} = H\begin{pmatrix} -T^{2H-1} & T^{2H-1} - |T-t_*|^{2H-1} \\ 0 & 0 \end{pmatrix},$$

$$A_{1,2} = H\begin{pmatrix} 0 & 0 \\ t_*^{2H-1} + |T-t_*|^{2H-1} & -t_*^{2H-1} \end{pmatrix},$$

$$A_{2,2} = H\left(H-\frac{1}{2}\right)\begin{pmatrix} 0 & 0 \\ t_*^{2H-2} + |T-t_*|^{2H-2} & -t_*^{2H-2} \end{pmatrix},$$

$$A_{5,1} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_{5,2} = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_{6,2,2} = 0.$$

Moreover,

$$\boldsymbol{w}(\boldsymbol{t}) = \Sigma^{-1}(\boldsymbol{t})\,\boldsymbol{b} = \frac{1}{t_1^{2H}t_2^{2H} - r^2(t_1,t_2)}\begin{pmatrix} t_2^{2H}\,a + r(t_1,t_2)\,b \\ r(t_1,t_2)\,a + t_1^{2H}\,b \end{pmatrix},$$

and

$$A_{1,1}\,\boldsymbol{w} \ne \boldsymbol{0}, \quad A_{1,2}\,\boldsymbol{w}(\boldsymbol{t}) \sim \boldsymbol{0}, \quad \boldsymbol{w}^\top A_{5,i}\,\boldsymbol{w} > 0, \quad \boldsymbol{w}^\top A_{2,i}\,\boldsymbol{w} > 0, \quad i = 1,2.$$

Note that $2 \in \mathcal{I}$. Since 2 is also the only element of $\mathcal{F}$, it follows that $\mathcal{F} \subset \mathcal{I}$, and we can use the second assertion of Corollary 2.1 instead of Theorem 2.1. Applying it with the data from Lemmata 2.3 and 2.4, we obtain the following result.

Theorem 2.3. *Let $a \ge b$ and set*

$$p(u) \coloneqq (1+\mathbb{1}_{a=b})\,\mathbb{P}\left\{B_H(T) > au,\ B_H(t_*) < -bu\right\}.$$

Then, with κ_1 and κ_2 from Lemma 2.3, we have the following results.

If $H < 1/2$,

$$\mathbb{P}\left\{\exists\, t,\, s \in [0,T] \colon B_H(t) > au,\ B_H(s) < -bu\right\} \sim \frac{9\,\pi\, w_1^{1/H}\, w_2^{1/H}\,\mathcal{H}^2}{2\,\kappa_1^{1/2}\,\kappa_2^{1/2}}\, u^{2/H-3}\, p(u),$$

where

$$\mathcal{H} = \lim_{S\to\infty}\frac{1}{S}\,\mathbb{E}\left\{\sup_{t\in[0,S]} e^{B_H(t)-t^{2H}/2}\right\}.$$

If $H = 1/2$,

$$\mathbb{P}\{\exists\, t,\, s \in [0,T] \colon B_H(t) > au,\ B_H(s) < -bu\} \sim \frac{3\sqrt{\pi}\, w_2^{1/H}\, \mathcal{H}\, \widetilde{\mathcal{H}}}{\kappa_2^{1/2}}\, u\, p(u)$$

where

$$\widetilde{\mathcal{H}} = \lim_{\Lambda\to\infty} \mathbb{E}\left\{ \sup_{t\in[0,\Lambda]} e^{B(t)-\lambda t} \right\} \in (0,\infty).$$

with[2]

$$\lambda = \frac{1}{2} + HT^{2H-1}w_1 - H\Big[T^{2H-1} + (T-t_*)^{2H-1}\Big]w_2$$

If $H > 1/2$,

$$\mathbb{P}\{\exists\, t,\, s \in [0,T] \colon B_H(t) > au,\ B_H(s) < -bu\} \sim \frac{3\sqrt{\pi}\, w_2^{1/H}\, \mathcal{H}}{\kappa_2^{1/2}}\, u^{1/H-1}\, p(u).$$

Remark 2.6. *Note that* $\kappa_2 \neq 2\,\boldsymbol{w}^\top A_{2,2}\,\boldsymbol{w}$*, as it were in [3, Remark 2.3, (2.15)]. Using Lemma 2.18, we can show that* $\kappa_2 = 2\,\boldsymbol{w}^\top A_{2,2}\,\boldsymbol{w} + \boldsymbol{w}^\top A_{1,2}\,\Sigma^{-1} A_{1,2}^\top\,\boldsymbol{w}$.

2.4 Auxiliary results

This Section consists of known results, taken from [3] and reproduced here for the reader's convenience.

2.4.1 Quadratic programming problem

For a given non-singular $d\times d$ real matrix Σ we consider the quadratic programming problem

$$\Pi_\Sigma(\boldsymbol{b})\colon \text{minimize } \boldsymbol{x}^\top\,\Sigma^{-1}\,\boldsymbol{x} \text{ under the linear constraint } \boldsymbol{x} \geq \boldsymbol{b}. \tag{2.17}$$

Below $J = \{1,\ldots,d\}\setminus I$ can be empty; the claim in (2.19) is formulated under the assumption that J is non-empty.

Lemma 2.5. *Let* $d \geq 2$ *and* Σ *a* $d\times d$ *symmetric positive definite matrix with inverse* Σ^{-1}*. If* $\boldsymbol{b} \in \mathbb{R}^d \setminus (-\infty,]^d$*, then* $\Pi_\Sigma(\boldsymbol{b})$ *has a unique solution* $\widetilde{\boldsymbol{b}}$ *and there exists a unique non-empty index set* $I \subset \{1,\ldots,d\}$ *with* $m \leq d$ *elements such that*

$$\widetilde{\boldsymbol{b}}_I = \boldsymbol{b}_I \neq \boldsymbol{0}_I \tag{2.18}$$

$$\widetilde{\boldsymbol{b}}_J = \Sigma_{JI}(\Sigma_{II})^{-1}\boldsymbol{b}_I \geq \boldsymbol{b}_J, \qquad (\Sigma_{II})^{-1}\boldsymbol{b}_I > \boldsymbol{0}_I, \tag{2.19}$$

$$\min_{\boldsymbol{x}\geq\boldsymbol{b}} \boldsymbol{x}^\top\,\Sigma^{-1}\,\boldsymbol{x} = \widetilde{\boldsymbol{b}}^\top\,\Sigma^{-1}\,\widetilde{\boldsymbol{b}} = \boldsymbol{b}_I^\top\,(\Sigma_{II})^{-1}\,\boldsymbol{b}_I > 0, \tag{2.20}$$

$$\max_{\boldsymbol{z}\in[0,\infty)^d\colon\, \boldsymbol{z}^\top\boldsymbol{b}>0} \frac{(\boldsymbol{z}^\top\boldsymbol{b})^2}{\boldsymbol{z}^\top\,\Sigma\,\boldsymbol{z}} = \frac{(\boldsymbol{w}^\top\boldsymbol{b})^2}{\boldsymbol{w}^\top\,\Sigma\,\boldsymbol{w}} = \min_{\boldsymbol{x}\geq\boldsymbol{b}} \boldsymbol{x}^\top\,\Sigma^{-1}\,\boldsymbol{x}, \tag{2.21}$$

with $\boldsymbol{w} = \Sigma^{-1}\,\widetilde{\boldsymbol{b}}$ *satisfying* $\boldsymbol{w}_I = (\Sigma_{II})^{-1}\boldsymbol{b}_I > \boldsymbol{0}_I$, $\boldsymbol{w}_J = \boldsymbol{0}_J$.

[2] It can be shown that $\lambda > 1/2$. Moreover, it may be represented as $\lambda = 1/2 + \kappa_1/w_1$.

Denote the solution map of the quadratic programming problem (2.17) by $\mathcal{P}\colon \Sigma^{-1} \mapsto \widetilde{\boldsymbol{b}}$ with $\widetilde{\boldsymbol{b}}$ the unique solution to $\Pi_\Sigma(\boldsymbol{b})$. The next result is a special case of [24, Theorem 3.1].

Lemma 2.6. *$\mathcal{P}$ is Lipshitz continuous on compact subset of the space of real $d \times d$ symmetric positive definite matrices.*

We will also need the following supplementary lemma on quadratic optimization.

Lemma 2.7. *Let E be a compact subset of $\mathbb{R}^n$ and $(\Sigma(\boldsymbol{t}))_{\boldsymbol{t}\in E}$ be a uniformly positive definite family of symmetric $d \times d$ matrices such that the map $\boldsymbol{t} \mapsto \Sigma(\boldsymbol{t})$ is continuous. Denote by $I(\boldsymbol{t})$, $K(\boldsymbol{t})$ and $L(\boldsymbol{t})$ the three index sets of $\Pi_{\Sigma(\boldsymbol{t})}(\boldsymbol{b})$, introduced in Lemma 2.5. Then the following assertions hold:*

1. *There exist three finite disjoint locally closed covers $(A_V)_{V\in 2^d}$, $(B_V)_{V\in 2^d}$ and $(C_{U,V})_{U,V\in 2^d}$ of E, such that*
$$I(\boldsymbol{t}) = \sum_{U\in 2^d} U^c\, \mathbb{1}_{A_U}(\boldsymbol{t}), \quad L(\boldsymbol{t}) = \sum_{V\in 2^d} V^c\, \mathbb{1}_{B_V}(\boldsymbol{t}), \quad K(\boldsymbol{t}) = \sum_{U,\,V\in 2^d} U\cap V\, \mathbb{1}_{C_{U,V}}(\boldsymbol{t}).$$
2. *The maps $\boldsymbol{t} \mapsto I(\boldsymbol{t})$ and $\boldsymbol{t} \mapsto L(\boldsymbol{t})$ are lower hemicontinuous. Moreover, for all $\boldsymbol{t} \in E$ there exists $\varepsilon(\boldsymbol{t}) > 0$ such that for all $\boldsymbol{s}$ such that $|\boldsymbol{t}-\boldsymbol{s}| < \varepsilon(\boldsymbol{t})$ holds $I(\boldsymbol{t}) \subset I(\boldsymbol{s})$.*

Remark 2.7. *Since $I(\boldsymbol{t})\cup K(\boldsymbol{t})\cup L(\boldsymbol{t}) = \{1,\dots,d\}$, it follows that the set-valued map $\boldsymbol{t} \mapsto K(\boldsymbol{t})$ is upper hemicontinuous.*

Remark 2.8. *Note that if the upper hemicontinuity property holds uniformly in $\boldsymbol{t}$, that is if $\varepsilon(\boldsymbol{t})$ can be taken independent of $\boldsymbol{t}$, then $\boldsymbol{t} \mapsto J(\boldsymbol{t})$ is constant.*

2.4.2 Borell-TIS and Piterbarg inequalities

Lemma 2.8. *Let $\boldsymbol{Z}(\boldsymbol{t})$, $\boldsymbol{t} \in E \subset \mathbb{R}^k$ be a separable centered d-dimensional vector-valued Gaussian random field having components with a.s. continuous paths. Assume that $\Sigma(\boldsymbol{t}) = \mathbb{E}\left\{\boldsymbol{Z}(\boldsymbol{t})\,\boldsymbol{Z}(\boldsymbol{t})^\top\right\}$ is non-singular for all $\boldsymbol{t} \in E$. Let $\boldsymbol{b} \in \mathbb{R}^d \setminus (-\infty,0]^d$ and define $\sigma_{\boldsymbol{b}}^2(\boldsymbol{t})$ as in (2.4). If $\sigma_{\boldsymbol{b}}^2 = \sup_{t\in E}\sigma_{\boldsymbol{b}^2}(\boldsymbol{t}) \in (0,\infty)$, then there exists some positive constant μ such that for all $u > \mu$*

$$\mathbb{P}\left\{\exists \boldsymbol{t}\in E\colon \boldsymbol{Z}(\boldsymbol{t}) > u\boldsymbol{b}\right\} \le \exp\left(-\frac{(u-\mu)^2}{2\sigma_{\boldsymbol{b}}^2}\right). \tag{2.22}$$

If further for some $C \in (0,\infty)$ and $\boldsymbol{\gamma} \in (0,2]^k$

$$\sum_{1\le i\le k} \mathbb{E}\left\{(Z_i(\boldsymbol{t}) - Z_i(\boldsymbol{s}))^2\right\} \le C\sum_{m=1}^{k} |t_m - s_m|^{\gamma_m} \tag{2.23}$$

and

$$\left\|\Sigma^{-1}(\boldsymbol{t}) - \Sigma^{-1}(\boldsymbol{s})\right\|_F \le C\sum_{m=1}^{k} |t_m - s_m|^{\gamma_m} \tag{2.24}$$

hold for all $\boldsymbol{t}$, $\boldsymbol{s} \in E$, then for all u positive

$$\mathbb{P}\left\{\exists \boldsymbol{t}\in E\colon \boldsymbol{Z}(\boldsymbol{t}) > u\boldsymbol{b}\right\} \le C_*\,\mathrm{mes}(E)\,u^{2d/\gamma-1}\exp\left(-\frac{u^2}{2\sigma_{\boldsymbol{b}}^2}\right), \tag{2.25}$$

where C_* is some positive constant not depending on u and $\gamma = \max_m \gamma_m$. In particular, if $\sigma_{\boldsymbol{b}}^2(\boldsymbol{t})$, $\boldsymbol{t} \in E$ is continuous and achieves its unique maximum at some fixed point $\boldsymbol{t}_* \in E$, then (2.25) is still valid if (2.23) and (2.24) are assumed to hold only for all $\boldsymbol{t}$, $\boldsymbol{s} \in E$ in an open neighborhood of $\boldsymbol{t}_*$.

2.4.3 Local Pickands lemma

Let us introduce the following assumptions.

B1 For all large u and all $\boldsymbol{\tau} \in Q_u$, the matrix $\Sigma_{u,\boldsymbol{\tau}} = R_{u,\boldsymbol{\tau}}(\mathbf{0},\mathbf{0})$ is positive definite and

$$\lim_{u\to\infty} \sup_{\boldsymbol{\tau}\in Q_u} u \left\| \Sigma - \Sigma_{u,\boldsymbol{\tau}} \right\|_F = 0 \tag{2.26}$$

holds for some positive definite matrix Σ.

B2 There exists a continuous $\mathbb{R}^d$-valued function $\boldsymbol{d}(\boldsymbol{t})$, $\boldsymbol{t} \in E$ and a continuous matrix-valued function $K(\boldsymbol{t},\boldsymbol{s})$, $(\boldsymbol{t},\boldsymbol{s}) \in E \times E$, such that

$$\lim_{u\to\infty} \sup_{\boldsymbol{\tau}\in Q_u,\ \boldsymbol{t}\in E} u \left\| \Sigma_{u,\boldsymbol{\tau}} - R_{u,\boldsymbol{\tau}}(\boldsymbol{t},\mathbf{0}) \right\|_F = 0, \tag{2.27}$$

$$\lim_{u\to\infty} \sup_{\boldsymbol{\tau}\in Q_u,\ \boldsymbol{t}\in \mathbb{E}} \left| u^2 \left[I - R_{u,\boldsymbol{\tau}}\, \Sigma_{u,\boldsymbol{\tau}}^{-1} \right] \widetilde{\boldsymbol{b}} - \boldsymbol{d}(\boldsymbol{t}) \right| = 0 \tag{2.28}$$

and

$$\lim_{u\to\infty} \sup_{\boldsymbol{\tau}\in Q_u} \sup_{\boldsymbol{t},\,\boldsymbol{s}\in E} \left\| u^2 \left[R_{u,\boldsymbol{\tau}}(\boldsymbol{t},\boldsymbol{s}) - R_{u,\boldsymbol{\tau}}(\boldsymbol{t},\mathbf{0})\, \Sigma_{u,\boldsymbol{\tau}}^{-1}\, R_{u,\boldsymbol{\tau}}(\mathbf{0},\boldsymbol{s}) \right] - K(\boldsymbol{t},\boldsymbol{s}) \right\|_F = 0. \tag{2.29}$$

B3 There exist positive constants C and $\boldsymbol{\gamma} \in (0,2]^k$ such that for any $\boldsymbol{t}$, $\boldsymbol{s} \in E$

$$\sup_{\boldsymbol{\tau}\in Q_u} u^2\, \mathbb{E}\left\{ \left| \boldsymbol{X}_{u,\boldsymbol{\tau}}(\boldsymbol{t}) - \boldsymbol{X}_{u,\boldsymbol{\tau}}(\boldsymbol{s}) \right|^2 \right\} \le C \sum_{m=1}^{k} |t_m - s_m|^{\gamma_m}. \tag{2.30}$$

For $\boldsymbol{Y}(\boldsymbol{t})$, $\boldsymbol{t} \in E$ a centered $\mathbb{R}^d$-valued Gaussian random field with a.s. continuous sample paths with cmf $K(\boldsymbol{s},\boldsymbol{t})$, $(\boldsymbol{s},\boldsymbol{t}) \in E \times E$ and an $\mathbb{R}^d$-valued function $\boldsymbol{d}$ define below

$$H_{\boldsymbol{Y},\boldsymbol{d}}(E) = \int_{\mathbb{R}^d} e^{\mathbf{1}^\top \boldsymbol{x}}\, \mathbb{P}\left\{ \exists \boldsymbol{t} \in E \colon \boldsymbol{Y}(\boldsymbol{t}) - \boldsymbol{d}(\boldsymbol{t}) > \boldsymbol{x} \right\} d\boldsymbol{x}\,. \tag{2.31}$$

Lemma 2.9. *Suppose that $\boldsymbol{X}_{u,\boldsymbol{\tau}}(\boldsymbol{t})$, $\boldsymbol{t} \in E$, $u > 0$, $\boldsymbol{\tau} \in Q_u$ satisfy B1, B2 and B3. Let $\boldsymbol{w} = \Sigma^{-1}\widetilde{\boldsymbol{b}}$, where $\widetilde{\boldsymbol{b}}$ is the unique solution of $\Pi_\Sigma(\boldsymbol{b})$. If $\boldsymbol{Y}_{\boldsymbol{w}}(\boldsymbol{t})$, $\boldsymbol{t} \in E$ has cmf $R(\boldsymbol{t},\boldsymbol{s}) = \mathrm{diag}(\boldsymbol{w})\, K(\boldsymbol{t},\boldsymbol{s})\, \mathrm{diag}(\boldsymbol{w})$ and $\boldsymbol{d}_{\boldsymbol{w}}(\boldsymbol{t}) = \mathrm{diag}(\boldsymbol{w})\, \boldsymbol{d}(\boldsymbol{t})$, then we have*

$$\lim_{u\to\infty} \sup_{\boldsymbol{\tau}\in Q_u} \left| \frac{\mathbb{P}\left\{ \exists \boldsymbol{t} \in E \colon \boldsymbol{X}_{u,\boldsymbol{\tau}}(\boldsymbol{t}) > u\boldsymbol{b} \right\}}{\mathbb{P}\left\{ \boldsymbol{X}_{u,\boldsymbol{\tau}}(\mathbf{0}) > u\boldsymbol{b} \right\}} - H_{\boldsymbol{Y}_{\boldsymbol{w}},\boldsymbol{d}_{\boldsymbol{w}}}(E) \right| = 0. \tag{2.32}$$

Remark 2.9. *If we suppose stronger assumptions on $\Sigma_{u,\boldsymbol{\tau}}$, for instance*

$$\lim_{u\to\infty} \sup_{\boldsymbol{\tau}\in Q_u} \left\| u^2 \left[\Sigma - \Sigma_{u,\boldsymbol{\tau}} \right] - \Xi \right\|_F = 0,$$

then as $u \to \infty$

$$\mathbb{P}\left\{ \boldsymbol{X}_{u,\boldsymbol{\tau}}(\mathbf{0}) > u\boldsymbol{b} \right\} \sim e^{-\boldsymbol{w}^\top \Xi\, \boldsymbol{w}/2}\, \mathbb{P}\left\{ \boldsymbol{\mathcal{N}} > u\boldsymbol{b} \right\},$$

where $\boldsymbol{\mathcal{N}}$ is a centered Gaussian vector with covariance matrix Σ.

2.4.4 Integral estimate

Lemma 2.10. *If a family of Hölder continuous Gaussian random fields $\boldsymbol{\chi}_{\boldsymbol{x}}(\boldsymbol{t})$, $\boldsymbol{t} \in [\boldsymbol{0}, \boldsymbol{\Lambda}]$ measurable in $\boldsymbol{x} \in \mathbb{R}^d$ satisfies*

$$\sup_{F \subset \{1,\ldots,d\}} \sup_{\boldsymbol{t} \in [\boldsymbol{0},\boldsymbol{\Lambda}]} \boldsymbol{w}_F^\top \mathbb{E}\left\{\boldsymbol{\chi}_{\boldsymbol{x},F}(\boldsymbol{t})\right\} \le G + \varepsilon \sum_{j=1}^d |x_j|$$

and

$$\sup_{F \subset \{1,\ldots,d\}} \sup_{\boldsymbol{t} \in [\boldsymbol{0},\boldsymbol{\Lambda}]} \operatorname{Var}\left\{\boldsymbol{w}_F^\top \boldsymbol{\chi}_{\boldsymbol{x},F}(\boldsymbol{t})\right\} \le \sigma^2$$

with some constants $\boldsymbol{w} > \boldsymbol{0}$, $\sigma^2 > 0$, $G \in \mathbb{R}$ and small enough $\varepsilon > 0$, then there exist constants C, $c > 0$ such that the following inequality holds:

$$\int_{\mathbb{R}^d} e^{\boldsymbol{w}^\top \boldsymbol{x}}\, \mathbb{P}\left\{\exists\, \boldsymbol{t} \in [\boldsymbol{0},\boldsymbol{\Lambda}] \colon \boldsymbol{\chi}_{\boldsymbol{x}}(\boldsymbol{t}) > \boldsymbol{x}\right\} d\boldsymbol{x} \le C e^{c(G+\sigma^2)}.$$

2.5 Proof of the main theorem

2.5.1 Log-layer bound

Lemma 2.11 (Log-layer bound)**.** *Suppose $\boldsymbol{X}$ satisfies Assumptions A1 to A3. Then there exist positive constants c, u_0 and $\boldsymbol{\Lambda}_0$ such that for $\boldsymbol{\Lambda} \ge \boldsymbol{\Lambda}_0$ and $u \ge u_0$*

$$\mathbb{P}\left\{\exists\, \boldsymbol{t} \in [\boldsymbol{0}, \boldsymbol{\delta}_u] \setminus u^{-2/\boldsymbol{\beta}}[\boldsymbol{0},\boldsymbol{\Lambda}] \colon \boldsymbol{X}(\boldsymbol{t}) > u\boldsymbol{b}\right\} \le c\, \mathbb{P}\left\{\boldsymbol{X}(\boldsymbol{0}) > u\boldsymbol{b}\right\} \exp\left(-\frac{1}{8}\sum_{i=1}^n \xi_i\, \Lambda_i^{\beta_i}\right),$$

where $\xi_i = \boldsymbol{w}^\top A_{2,i}\, \boldsymbol{w} > 0$ by A2.4.

Proof. For simplicity assume that $I = \{1, \ldots, d\}$, hence $\widetilde{\boldsymbol{b}} = \boldsymbol{b}$. The idea of the proof is to split the log-layer $[\boldsymbol{0}, u^{-2/\boldsymbol{\beta}} \ln^{2/\boldsymbol{\beta}} u] \setminus u^{-2/\boldsymbol{\beta}}[\boldsymbol{0},\boldsymbol{\Lambda}]$ into tiny pieces

$$\boldsymbol{\Lambda} u^{-2/\boldsymbol{\nu}}[\boldsymbol{k}, \boldsymbol{k}+\boldsymbol{1}], \qquad \boldsymbol{k} \in Q_u = \bigcup_{\mathcal{L} \ne \varnothing} Q_u(\mathcal{L}),$$

where the union is taken over non-empty subsets $\mathcal{L}$ of $\{1, \ldots, n\}$ and

$$Q_u(\mathcal{L}) = \left\{\boldsymbol{k} \in \mathbb{Z}_+^n \colon k_i \ge u^{2/\nu_i - 2/\beta_i}/\Lambda_i, \quad i \in \mathcal{L}\right\} \cap \left[\boldsymbol{0}, u^{2/\boldsymbol{\nu} - 2/\boldsymbol{\beta}}/\boldsymbol{\Lambda}\right].$$

Next, derive a suitable *uniform* in $\boldsymbol{k} \in Q_u$ bound for the Pickands intervals' probabilities

$$\mathbb{P}\left\{\exists\, \boldsymbol{t} \in \boldsymbol{\Lambda} u^{-2/\boldsymbol{\nu}}[\boldsymbol{k}, \boldsymbol{k}+\boldsymbol{1}] \colon \boldsymbol{X}(\boldsymbol{t}) > u\boldsymbol{b}\right\}, \tag{2.33}$$

and then sum them up to obtain an upper bound on the layer's probability. To this end, let us define a family of random fields

$$\boldsymbol{X}_{u,\boldsymbol{k}}(\boldsymbol{t}) = \boldsymbol{X}\left(u^{-2/\boldsymbol{\nu}}\left(\boldsymbol{\Lambda}\boldsymbol{k} + \boldsymbol{t}\right)\right), \qquad \boldsymbol{k} \in Q_u, \quad \boldsymbol{t} \in [\boldsymbol{0}, \boldsymbol{\Lambda}],$$

and denote the corresonding covariance and variance matrices by

$$R_{u,\boldsymbol{k}}(\boldsymbol{t}, \boldsymbol{s}) = R\left(u^{-2/\boldsymbol{\nu}}\left(\boldsymbol{\Lambda}\boldsymbol{k} + \boldsymbol{t}\right), u^{-2/\boldsymbol{\nu}}\left(\boldsymbol{\Lambda}\boldsymbol{k} + \boldsymbol{s}\right)\right), \qquad \Sigma_{u,\boldsymbol{k}} = \Sigma\left(\boldsymbol{\Lambda} u^{-2/\boldsymbol{\nu}} \boldsymbol{k}\right).$$

Next, apply the law of total probability

$$\mathbb{P}\left\{\exists \boldsymbol{t} \in \boldsymbol{\Lambda} u^{-2/\boldsymbol{\nu}}\, [\boldsymbol{k}, \boldsymbol{k}+\boldsymbol{1}] : \boldsymbol{X}(\boldsymbol{t}) > u\boldsymbol{b}\right\}$$
$$= u^{-d} \int_{\mathbb{R}^d} \mathbb{P}\left\{\exists \boldsymbol{t} \in [\boldsymbol{0}, \boldsymbol{\Lambda}] \colon \boldsymbol{\chi}_{u,\boldsymbol{k}}(\boldsymbol{t}) > \boldsymbol{x}\right\} \varphi_{\Sigma_{u,\boldsymbol{k}}}\left(u\boldsymbol{b} - \frac{\boldsymbol{x}}{u}\right) d\boldsymbol{x}, \tag{2.34}$$

where $\boldsymbol{\chi}_{u,\boldsymbol{k}}(\boldsymbol{t})$ denotes the conditional process $u\,(\boldsymbol{X}_{u,\boldsymbol{k}}(\boldsymbol{t}) - u\boldsymbol{b}) + \boldsymbol{x}$ given $\boldsymbol{X}_{u,\boldsymbol{k}}(\boldsymbol{0}) = u\boldsymbol{b} - u^{-1}\boldsymbol{x}$. First, bound $\varphi_{\Sigma_{u,\boldsymbol{k}}}$ using (2.77)

$$\ln\left(\frac{\varphi_{\Sigma_{u,\boldsymbol{k}}}\left(u\boldsymbol{b} - \frac{\boldsymbol{x}}{u}\right)}{\varphi_{\Sigma}(u\boldsymbol{b})}\right) \le -\frac{1}{2}\, u^2\, \boldsymbol{b}^\top \left[\Sigma_{u,\boldsymbol{k}}^{-1} - \Sigma^{-1}\right] \boldsymbol{b} + \boldsymbol{b}^\top \Sigma_{u,k}^{-1}\, \boldsymbol{x}.$$

Plugging this into (2.34) and noting that $u^{-d}\, \varphi_{\Sigma}(u\boldsymbol{b}) \sim \mathbb{P}\left\{\boldsymbol{X}(\boldsymbol{0}) > u\boldsymbol{b}\right\}$, we obtain the following bound:

$$\mathbb{P}\left\{\exists \boldsymbol{t} \in \boldsymbol{\Lambda} u^{-2/\boldsymbol{\nu}}\, [\boldsymbol{k}, \boldsymbol{k}+1] : \boldsymbol{X}(\boldsymbol{t}) > u\boldsymbol{b}\right\} \le AB\, \mathbb{P}\left\{\boldsymbol{X}(\boldsymbol{0}) > u\boldsymbol{b}\right\}, \tag{2.35}$$

where

$$A \coloneqq \exp\left(-\frac{1}{2}\, u^2\, \boldsymbol{b}^\top \left[\Sigma_{u,\boldsymbol{k}}^{-1} - \Sigma^{-1}\right] \boldsymbol{b}\right),$$

$$B \coloneqq \int_{\mathbb{R}^d} e^{\boldsymbol{b}^\top \Sigma_{u,\boldsymbol{k}}^{-1} \boldsymbol{x}}\, \mathbb{P}\left\{\exists \boldsymbol{t} \in [\boldsymbol{0}, \boldsymbol{\Lambda}] \colon \boldsymbol{\chi}_{u,\boldsymbol{k}}(\boldsymbol{t}) > \boldsymbol{x}\right\} d\boldsymbol{x}\,.$$

At this point, we split the proof in four parts: bounding A, bounding B, comparing the bounds and summing them in $\boldsymbol{k}$.

Bounding A. By (2.79), we have

$$\boldsymbol{b}^\top \left[\Sigma^{-1}(\boldsymbol{\tau}) - \Sigma^{-1}\right] \boldsymbol{b} = \sum_{i,j=1}^{n} \Xi_{i,j}\, \tau_i^{\beta_i/2}\, \tau_j^{\beta_j/2} + o\left(\sum_{i=1}^{n} \tau_i^{\beta_i}\right), \tag{2.36}$$

where

$$\Xi_{i,j} = \boldsymbol{w}^\top \widetilde{D}_{i,j}\, \boldsymbol{w}, \qquad \widetilde{D}_{i,j} = 2\, A_{2,i}\, \mathbb{1}_{i=j} + \left[A_{6,i,j} + A_{1,i} \Sigma^{-1} A_{1,j}^\top\right] \mathbb{1}_{i,j \in \mathcal{F}},$$

Using (2.82), we can bound (2.36) for $\boldsymbol{\tau}$ close enough to $\boldsymbol{0}$ by

$$\boldsymbol{b}^\top \left[\Sigma^{-1}(\boldsymbol{\tau}) - \Sigma^{-1}\right] \boldsymbol{b} = \sum_{i,j=1}^{n} \Xi_{i,j}\, \tau_i^{\beta_i/2}\, \tau_j^{\beta_j/2} + o\left(\sum_{i=1}^{n} \tau_i^{\beta_i}\right) \ge \frac{3}{2} \sum_{i=1}^{n} \boldsymbol{w}^\top A_{2,i}\, \boldsymbol{w}\, \tau_i^{\beta_i}.$$

Plugging $\boldsymbol{\tau} = u^{-2/\boldsymbol{\nu}} \boldsymbol{\Lambda} \boldsymbol{k}$, we obtain that for all large enough u holds

$$-\frac{1}{2}\, u^2\, \boldsymbol{b}^\top \left[\Sigma_{u,\boldsymbol{k}}^{-1} - \Sigma^{-1}\right] \boldsymbol{b} \le -\frac{3}{4} \sum_{i=1}^{n} \boldsymbol{w}^\top A_{2,i}\, \boldsymbol{w}\ u^{2-2\beta_i/\nu_i}\, (\Lambda_i k_i)^{\beta_i}. \tag{2.37}$$

Bounding B. As a next step, we show that

$$\int_{\mathbb{R}^d} e^{\boldsymbol{b}^\top \Sigma_{u,\boldsymbol{k}}^{-1} \boldsymbol{x}}\, \mathbb{P}\left\{\exists \boldsymbol{t} \in [\boldsymbol{0}, \boldsymbol{\Lambda}] \colon \boldsymbol{\chi}_{u,\boldsymbol{k}}(\boldsymbol{t}) > \boldsymbol{x}\right\} d\boldsymbol{x} \le c_1 e^{c_2 (G + \sigma^2)} \tag{2.38}$$

using Lemma 2.10. Here $G \in \mathbb{R}$ and $\sigma^2 > 0$ are any such numbers that for all small enough ε hold

$$\sup_{F \subset \{1,\dots,d\}} \sup_{\boldsymbol{t} \in [\boldsymbol{0}, \boldsymbol{\Lambda}]} \boldsymbol{w}_F^\top\, \mathbb{E}\left\{\boldsymbol{\chi}_{u,\boldsymbol{k},F}(\boldsymbol{t})\right\} \le G + \varepsilon \sum_{j=1}^{d} |x_j|,$$
$$\sup_{F \subset \{1,\dots,d\}} \sup_{\boldsymbol{t} \in [\boldsymbol{0}, \boldsymbol{\Lambda}]} \operatorname{Var}\left\{\boldsymbol{w}_F^\top\, \boldsymbol{\chi}_{u,\boldsymbol{k},F}(\boldsymbol{t})\right\} \le \sigma^2. \tag{2.39}$$

For our current needs they also must be uniform in $\boldsymbol{k}$. The following estimate for $G + \sigma^2$ is proven in the Appendix (see Section 2.6.4)

$$G+\sigma^2 = c_5 \sum_{i=1}^{n} \left[u^{2-2\beta_i/\nu_i} \Lambda_i^{\beta_i} \Big[(k_i \vee 1)^{\beta_i - 1} + \varepsilon^{-1} (k_i \vee 1)^{\beta_i - 2} + \varepsilon \left(k_i^{\beta_i} + 1 \right) \Big] + u^{2-2\alpha_i/\nu_i} \Lambda_i^{\alpha_i} \right]. \quad (2.40)$$

Comparing the bounds. Combining the bound (2.37) for A, the bound (2.38) for B with the $G + \sigma^2$ estimated in (2.40), and plugging all this into the AB bound (2.35), we arrive at the following inequality:

$$\ln \left(\frac{\mathbb{P}\left\{ \exists\, \boldsymbol{t} \in \boldsymbol{\Lambda} u^{-2/\boldsymbol{\nu}} [\boldsymbol{k}, \boldsymbol{k}+\boldsymbol{1}] : \boldsymbol{X}(\boldsymbol{t}) > u\boldsymbol{b} \right\}}{c_1\, \mathbb{P}\left\{ \boldsymbol{X}(\boldsymbol{0}) > u\boldsymbol{b} \right\}} \right)$$

$$\leq -\frac{3}{4} \sum_{i=1}^{n} \boldsymbol{w}^\top A_{2,i}\, \boldsymbol{w}\, u^{2-2\beta_i/\nu_i} (\Lambda_i k_i)^{\beta_i} + c_6 \sum_{i=1}^{n} \Bigg[u^{2-2\beta_i/\nu_i} \Lambda_i^{\beta_i} \Big[(k_i \vee 1)^{\beta_i - 1}$$

$$+ \varepsilon^{-1} (k_i \vee 1)^{\beta_i - 2} + \varepsilon \left(k_i^{\beta_i} + 1 \right) \Big] + u^{2-2\alpha_i/\nu_i} \Lambda_i^{\alpha_i} \Bigg]. \quad (2.41)$$

Setting $\varepsilon = \boldsymbol{w}^\top A_{2,i}\, \boldsymbol{w} / 4\, c_6$, we find that the i-th term is at most

$$u^{2-2\beta_i/\nu_i} \Lambda_i^{\beta_i} \left[-\frac{\boldsymbol{w}^\top A_{2,i}\, \boldsymbol{w}}{2} k_i^{\beta_i} + c_6 \Big[(k_i \vee 1)^{\beta_i - 1} + \varepsilon^{-1} (k_i \vee 1)^{\beta_i - 2} + \varepsilon \Big] \right] + c_6\, u^{2-2\alpha_i/\nu_i} \Lambda_i^{\alpha_i}. \quad (2.42)$$

Note that if $k_i = 0$, which is only possible if $i \in \mathcal{L}^c$, the equation (2.42) reads:

$$c_7\, u^{2-2\beta_i/\nu_i} \Lambda_i^{\beta_i} + c_6\, u^{2-2\alpha_i/\nu_i} \Lambda_i^{\alpha_i}.$$

If $k_i \geq 1$, then $k_i \vee 1 = k_i$, and (2.42) is at most

$$-\frac{\boldsymbol{w}^\top A_{2,i}\, \boldsymbol{w}}{4} k_i^{\beta_i} u^{2-2\beta_i/\nu_i} \Lambda_i^{\beta_i} + c_6\, u^{2-2\alpha_i/\nu_i} \Lambda_i^{\alpha_i} \quad (2.43)$$

under the following conditions:

1. if $i \in \mathcal{L} \cap \mathcal{I}$, $k_i \geq u^{2/\alpha_i - 2/\beta_i} / \Lambda_i \to \infty$, and we can choose u_0 such that (2.43) holds for all; $k_i \geq 1$ and $u \geq u_0$
2. if $i \in \mathcal{L}^c \cup \mathcal{I}^c$, there exists $k_{0,i}$, independent of u, such that (2.43) holds for $k_i \geq k_{0,i}$.

Combined upper bound for $\boldsymbol{k} \geq \boldsymbol{k}_0$. Denote

$$k_{0,i} = k_{0,i} \vee 1 \quad \text{for} \quad i \in \mathcal{I}^c \quad \text{and} \quad k_{0,i} = u^{2/\nu_i - 2/\beta_i} \quad \text{for} \quad i \in \mathcal{L} \cap \mathcal{I}.$$

We have shown that there exits u_0 and such that if $u \geq u_0$ and $\boldsymbol{k} \geq \boldsymbol{k}_0$, then the left-hand side of (2.41) is at most

$$-\frac{1}{4} \sum_{i \in \mathcal{L} \cup \mathcal{I}}^{n} \left[u^{2-2\beta_i/\nu_i} \Lambda_i^{\beta_i}\, \boldsymbol{w}^\top A_{2,i}\, \boldsymbol{w}\, k_i^{\beta_i} + c_6\, u^{2-2\alpha_i/\nu_i} \Lambda_i^{\alpha_i} \right] + c_8.$$

Hence,

$$
\frac{\mathbb{P}\left\{\exists\, \boldsymbol{t} \in \boldsymbol{\Lambda} u^{-2/\boldsymbol{\nu}}\, [\boldsymbol{k}, \boldsymbol{k}+\boldsymbol{1}] : \boldsymbol{X}(\boldsymbol{t}) > u\boldsymbol{b}\right\}}{c_9\, \mathbb{P}\left\{\boldsymbol{X}(\boldsymbol{0}) > u\boldsymbol{b}\right\}}
\leq \prod_{i=1}^{n} \exp\left(-\frac{\boldsymbol{w}^\top A_{2,i} \boldsymbol{w}}{4}\, u^{2-2\beta_i/\nu_i}\, (\Lambda_i k_i)^{\beta_i} + c_3 u^{2-2\alpha_i/\nu_i}\, \Lambda_i^{\alpha_i} \right). \tag{2.44}
$$

Let us now show how to cover $\boldsymbol{k}$'s with $1 \leq k_i \leq k_{0,i}$, $i \in \mathcal{I}^c$. Assume, for the sake of simplicity, that $\boldsymbol{k}$ is such that there is exactly one i such that $1 \leq k_i \leq k_{0,i}$. The general case can be addressed in a similar way. Note that so far we did not assume anything about Λ_i except positivity. We want to exploit this fact. To this end, set $\widetilde{\Lambda}_i := \Lambda_i / k_{0,i}$ and $\boldsymbol{x}' = (x_j)_{j \neq i}$. By (2.44) we have that

$$
\mathbb{P}\left\{\exists\, \boldsymbol{t} \in \boldsymbol{\Lambda} u^{-2/\boldsymbol{\nu}}\, [\boldsymbol{k}, \boldsymbol{k}+\boldsymbol{1}] : \boldsymbol{X}(\boldsymbol{t}) > u\boldsymbol{b}\right\}
\leq \sum_{j=k_{0,i}k_i}^{k_{0,i}(k_i+1)-1} \mathbb{P}\left\{ \begin{matrix} \exists\, t_i \in \widetilde{\Lambda}_i\, u^{-2/\nu_i}[j, j+1] \\ \exists\, \boldsymbol{t}' \in \boldsymbol{\Lambda}' u^{-2/\boldsymbol{\nu}'_i}[\boldsymbol{k}', \boldsymbol{k}'+\boldsymbol{1}'] \end{matrix} \quad : \quad \boldsymbol{X}(\boldsymbol{t}) > u\boldsymbol{b} \right\}
$$

Applying (2.44) to the summands, we find that the same bound (2.44) holds true in this case, but with a different constant.

Bound improvement for $k_i = 0$. Allowing $k_i = 0$ for some $i \in \mathcal{L}^c$, we obtain

$$
\begin{aligned}
\frac{\mathbb{P}\left\{\exists\, \boldsymbol{t} \in \boldsymbol{\Lambda} u^{-2/\boldsymbol{\nu}}\, [\boldsymbol{k}, \boldsymbol{k}+\boldsymbol{1}] : \boldsymbol{X}(\boldsymbol{t}) > u\boldsymbol{b}\right\}}{c_9\, \mathbb{P}\left\{\boldsymbol{X}(\boldsymbol{0}) > u\boldsymbol{b}\right\}}
&\leq \prod_{i=1}^{n} \exp\left(-\frac{\boldsymbol{w}^\top A_{2,i} \boldsymbol{w}}{4}\, u^{2-2\beta_i/\nu_i}\, (\Lambda_i k_i)^{\beta_i} + c_3 u^{2-2\alpha_i/\nu_i}\, \Lambda_i^{\alpha_i} \right) \\
&\quad \times \prod_{j \in \mathcal{L}^c\colon\, k_j = 0} \exp\left(c_6 u^{2-2\beta_j/\nu_j} \Lambda_j^{\beta_j} \right).
\end{aligned}
$$

Note that if $j \in \mathcal{L}^c \cap \mathcal{I}$, then the corresponding factor is bounded by a constant, since $u^{2-2\beta_j/\nu_j} \to 0$. If $i \in \mathcal{I}^c$, set $\widetilde{\Lambda}_i = 1$ and apply the same trick as above. That is, slice in the i-th direction and sum back:

$$
\mathbb{P}\left\{\exists\, \boldsymbol{t} \in \boldsymbol{\Lambda} u^{-2/\boldsymbol{\nu}}\, [\boldsymbol{k}, \boldsymbol{k}+\boldsymbol{1}] : \boldsymbol{X}(\boldsymbol{t}) > u\boldsymbol{b}\right\}
\leq \sum_{j=0}^{\Lambda_j+1} \mathbb{P}\left\{ \begin{matrix} \exists\, t_i \in\, u^{-2/\nu_i}[j, j+1] \\ \exists\, \boldsymbol{t}' \in \boldsymbol{\Lambda}' u^{-2/\boldsymbol{\nu}'_i}[\boldsymbol{k}', \boldsymbol{k}'+\boldsymbol{1}'] \end{matrix} \quad : \quad \boldsymbol{X}(\boldsymbol{t}) > u\boldsymbol{b} \right\}.
$$

Since $\widetilde{\Lambda}_i = 1$, the product in $j \in \mathcal{L}^c\colon k_j = 0$ becomes a constant. Other factors remain the same, except for the i-th, which gives

$$
\sum_{j=0}^{\Lambda_i} \exp\left(-\frac{\boldsymbol{w}^\top A_{2,i}\, \boldsymbol{w}}{4}\, j^{\beta_i} \right) \leq c_{10}\, \exp\left(-\frac{\boldsymbol{w}^\top A_{2,i}\, \boldsymbol{w}}{4} \right) = c_{11}.
$$

Summing up the bounds in $\boldsymbol{k}$. Summing (2.44) in $\boldsymbol{k}$, we obtain

$$\sum_{\boldsymbol{k}\in Q_u(\mathcal{L})} \frac{\mathbb{P}\left\{\exists\, \boldsymbol{t}\in \boldsymbol{\Lambda} u^{-2/\boldsymbol{\nu}}[\boldsymbol{k},\boldsymbol{k}+\boldsymbol{1}]\colon \boldsymbol{X}(\boldsymbol{t}) > u\boldsymbol{b}\right\}}{c_{12}\,\mathbb{P}\left\{\boldsymbol{X}(\boldsymbol{0}) > u\boldsymbol{b}\right\}}$$

$$\le \prod_{i\in\mathcal{I}} \sum_{k_i\neq 0} \exp\left(-\frac{\boldsymbol{w}^\top A_{2,i}\,\boldsymbol{w}}{4}\, u^{2-2\beta_i/\nu_i}\,(\Lambda_i k_i)^{\beta_i} + c_3\,\Lambda_i^{\alpha_i}\right)$$

$$\times \prod_{j\in\mathcal{I}^c} \sum_{k_j\neq 0} \exp\left(-\frac{\boldsymbol{w}^\top A_{2,j}\,\boldsymbol{w}}{4}\,(\Lambda_j k_j)^{\beta_j} + c_3 u^{2-2\alpha_j/\beta_j}\,\Lambda_j^{\alpha_j}\right).$$

If $i\in\mathcal{I}$, then

$$\sum_{k_i\ge u^{2/\alpha_i-2/\beta_i}} \exp\left(-\frac{1}{4}\,u^{2-2\beta_i/\nu_i}\,\boldsymbol{w}^\top A_{2,i}\,\boldsymbol{w}\,(\Lambda_i k_i)^{\beta_i} + c_3\,\Lambda_i^{\alpha_i}\right)$$

$$\le c_5 \exp\left(-\frac{\boldsymbol{w}^\top A_{2,i}\,\boldsymbol{w}}{4}\,\Lambda_i^{\beta_i} + c_3\,\Lambda_i^{\alpha_i}\right) \le c_5 \exp\left(-\frac{1}{8}\,\boldsymbol{w}^\top A_{2,i}\,\boldsymbol{w}\,\Lambda_i^{\beta_i}\right),$$

where the last inequality is true for $\Lambda_i \ge \Lambda_{0,i}$ with $\Lambda_{0,i}^{\beta_i-\alpha_i}\,\boldsymbol{w}^\top A_{2,i}\,\boldsymbol{w} \ge 8\,c_3$. Same upper bound works for $i\in\mathcal{I}^c$. Indeed, $u^{2-2\alpha_j/\beta_j}\Lambda_j^{\alpha_j}$ may be bounded by a constant and

$$\sum_{k_j\neq 0} \exp\left(-\frac{\boldsymbol{w}^\top A_{2,j}\boldsymbol{w}}{4}\,(\Lambda_j k_j)^{\beta_j}\right) \le \exp\left(-\frac{\boldsymbol{w}^\top A_{2,j}\boldsymbol{w}}{8}\,\Lambda_j^{\beta_j}\right).$$

□

Remark 2.10. *Note that $\boldsymbol{w}^\top A_{2,i}\,\boldsymbol{w} > 0$ has to be satisfied for all i's, because otherwise one of the sums in k_i (as, for example, the last sum of the proof) may be infinite.*

2.5.2 Double sum bound

Define for $\boldsymbol{\tau},\boldsymbol{\lambda},\boldsymbol{S}\in\mathbb{R}_+^n$ the double events' probabilities by

$$P_{\boldsymbol{b}}(\boldsymbol{\tau},\boldsymbol{\lambda},\boldsymbol{S}) := \mathbb{P}\left\{\begin{matrix}\exists\,\boldsymbol{t}\in u^{-2/\boldsymbol{\nu}}[\boldsymbol{\tau},\boldsymbol{\tau}+\boldsymbol{S}]\colon & \boldsymbol{X}(\boldsymbol{t}) > u\boldsymbol{b},\\ \exists\,\boldsymbol{s}\in u^{-2/\boldsymbol{\nu}}[\boldsymbol{\lambda},\boldsymbol{\lambda}+\boldsymbol{S}]\colon & \boldsymbol{X}(\boldsymbol{s}) > u\boldsymbol{b}\end{matrix}\right\}. \tag{2.45}$$

Lemma 2.12 (Double sum bound)**.** *Let double events' displacements $\boldsymbol{\tau}$, $\boldsymbol{\lambda}\in[\boldsymbol{0},\boldsymbol{N}_u]$ be such that*

1. There is no offset in Piterbarg and Talagrand type coordinates: $\boldsymbol{\tau}_{\mathcal{J}\cup\mathcal{K}} = \boldsymbol{\lambda}_{\mathcal{J}\cup\mathcal{K}} = \boldsymbol{0}_{\mathcal{J}\cup\mathcal{K}}$.

2. There exists a (possibly empty) subset of Pickands type coordinates $\mathcal{I}'\subset\mathcal{I}$ in which there is no offset: $\boldsymbol{\tau}_{\mathcal{I}'} = \boldsymbol{\lambda}_{\mathcal{I}'}$.

3. The offset in the remaining Pickands type coordinates $\mathcal{I}_2 := \mathcal{I}\setminus\mathcal{I}'$ is strictly greater than $\boldsymbol{S}$: $\boldsymbol{\lambda}_{\mathcal{I}_2} - \boldsymbol{\tau}_{\mathcal{I}_2} > \boldsymbol{S}_{\mathcal{I}_2}$.

Then, there exists $u_0\ge 0$ and $\boldsymbol{S}_0 > \boldsymbol{0}$ such that for all $u\ge u_0$ and $\boldsymbol{S}\ge\boldsymbol{S}_0$ holds

$$\frac{P_{\boldsymbol{b}}(\boldsymbol{\tau},\boldsymbol{\lambda},\boldsymbol{S})}{H_u(\boldsymbol{\tau})\,\mathbb{P}\left\{\boldsymbol{X}(\boldsymbol{0}) > u\boldsymbol{b}\right\}} \le C \prod_{i\in\mathcal{J}\cup\mathcal{K}} e^{cS_i^{\beta_i}} \prod_{i\in\mathcal{I}'} S_i \prod_{i\in\mathcal{I}_2}\left[\frac{S_i}{\lambda_i-\tau_i-S_i}\right]^2 \exp\left(-\frac{\varkappa_i}{64}\,(\lambda_i-\tau_i-S_i)^{\alpha_i}\right)$$

with some constants $C,\ c > 0$ and

$$H_u(\boldsymbol{\tau}) = \exp\left(-\frac{1}{4} \sum_{i \in \mathcal{I}_2} \xi_i \, \tau_i^{\beta_i} u^{2-2\beta_i/\nu_i} \right).$$

Remark 2.11. *We want to stress the fact that the conditions of the lemma demand that there be no Pickands type coordinates $i \in \mathcal{I}$ with offsets smaller than S_i, except those in which the offset is zero. This is not a coincidence, since the adjacent intervals are to be dealt with differently (see proof of Theorem 2.1 for details). Note also that if $\mathcal{I}' = \mathcal{I}$, the assertion of the lemma is trivial.*

Proof. For simplicity assume that $I = \{1, \ldots, d\}$, hene $\widetilde{\boldsymbol{b}} = \boldsymbol{b}$. Next, note that if $\boldsymbol{X}(\boldsymbol{t})$ and $\boldsymbol{X}(\boldsymbol{s})$ exceed $u\boldsymbol{b}$, then their sum exceeds $2u\boldsymbol{b}$:

$$P_{\boldsymbol{b}}(\boldsymbol{\tau}, \boldsymbol{\lambda}, \boldsymbol{S}) \le \mathbb{P}\left\{ \exists\, \boldsymbol{t} \in u^{-2/\boldsymbol{\nu}}[\boldsymbol{\tau}, \boldsymbol{\tau} + \boldsymbol{S}],\ \exists\, \boldsymbol{s} \in u^{-2/\boldsymbol{\nu}}[\boldsymbol{\lambda}, \boldsymbol{\lambda} + \boldsymbol{S}] \colon \boldsymbol{X}(\boldsymbol{t}) + \boldsymbol{X}(\boldsymbol{s}) > 2u\boldsymbol{b} \right\}$$
$$= \mathbb{P}\left\{ \exists\, \boldsymbol{t}, \boldsymbol{s} \in [\boldsymbol{0}, \boldsymbol{S}] \colon \boldsymbol{X}_{u,\boldsymbol{\tau},\boldsymbol{\lambda}}(\boldsymbol{t}, \boldsymbol{s}) > u\boldsymbol{b} \right\}$$

where

$$\boldsymbol{X}_{u,\boldsymbol{\tau},\boldsymbol{\lambda}}(\boldsymbol{t}, \boldsymbol{s}) = \frac{1}{2} \left(\boldsymbol{X}(u^{-2/\boldsymbol{\nu}}\boldsymbol{\tau} + u^{-2/\boldsymbol{\nu}}\boldsymbol{t}) + \boldsymbol{X}(u^{-2/\boldsymbol{\nu}}\boldsymbol{\lambda} + u^{-2/\boldsymbol{\nu}}\boldsymbol{s}) \right).$$

Henceforth, we shall seek a bound of the latter probability. To this end, we employ an idea analogous to that of the proof of Lemma 2.11: first, apply the law of total probability

$$\begin{aligned} &\mathbb{P}\left\{ \exists\, \boldsymbol{t},\ \boldsymbol{s} \in [\boldsymbol{0}, \boldsymbol{S}] \colon \boldsymbol{X}_{u,\boldsymbol{\tau},\boldsymbol{\lambda}}(\boldsymbol{t}, \boldsymbol{s}) > u\boldsymbol{b} \right\} \\ &\qquad = u^{-d} \int_{\mathbb{R}^d} \mathbb{P}\left\{ \exists\, \boldsymbol{t}, \boldsymbol{s} \in [\boldsymbol{0}, \boldsymbol{S}] \colon \chi_{u,\boldsymbol{\tau},\boldsymbol{\lambda}}(\boldsymbol{t}, \boldsymbol{s}) > \boldsymbol{x} \right\} \varphi_{\Sigma_u(\boldsymbol{\tau},\boldsymbol{\lambda})}\left(u\boldsymbol{b} - \frac{\boldsymbol{x}}{u} \right) d\boldsymbol{x}, \end{aligned} \tag{2.46}$$

where the conditional random field is defined by

$$\chi_{u,\boldsymbol{\tau},\boldsymbol{\lambda}}(\boldsymbol{t}, \boldsymbol{s}) = u \left(\boldsymbol{X}_{u,\boldsymbol{\tau},\boldsymbol{\lambda}}(\boldsymbol{t}, \boldsymbol{s}) - \boldsymbol{X}_{u,\boldsymbol{\tau},\boldsymbol{\lambda}}(\boldsymbol{0}, \boldsymbol{0}) \;\middle|\; \boldsymbol{X}_{u,\boldsymbol{\tau},\boldsymbol{\lambda}}(\boldsymbol{0}, \boldsymbol{0}) = u\boldsymbol{b} - \frac{\boldsymbol{x}}{u} \right)$$

and $\Sigma_u(\boldsymbol{\tau}, \boldsymbol{\lambda})$ is the variance matrix:

$$\Sigma_u(\boldsymbol{\tau}, \boldsymbol{\lambda}) = \mathbb{E}\left\{ \boldsymbol{X}_{u,\boldsymbol{\tau},\boldsymbol{\lambda}}(\boldsymbol{0}, \boldsymbol{0})\, \boldsymbol{X}_{u,\boldsymbol{\tau},\boldsymbol{\lambda}}(\boldsymbol{0}, \boldsymbol{0})^\top \right\}.$$

Next, bounding the exponential prefactor similarly to (2.77) as follows:

$$\ln\left(\frac{\varphi_{\Sigma_u(\boldsymbol{\tau},\boldsymbol{\lambda})}(u\boldsymbol{b} - u^{-1}\boldsymbol{x})}{\varphi_\Sigma(u\boldsymbol{b})} \right) \le -\frac{1}{2} u^2 \boldsymbol{b}^\top \left[\Sigma_u^{-1}(\boldsymbol{\tau}, \boldsymbol{\lambda}) - \Sigma^{-1} \right] \boldsymbol{b} + \boldsymbol{b}^\top \Sigma_u^{-1}(\boldsymbol{\tau}, \boldsymbol{\lambda})\, \boldsymbol{x},$$

and using

$$\mathbb{P}\{\boldsymbol{X}(\boldsymbol{0}) > u\boldsymbol{b}\} \sim u^{-d} \varphi_\Sigma(u\boldsymbol{b}),$$

we obtain

$$\mathbb{P}\left\{ \exists\, \boldsymbol{t},\ \boldsymbol{s} \in [\boldsymbol{0}, \boldsymbol{S}] \colon \boldsymbol{X}_{u,\boldsymbol{\tau},\boldsymbol{\lambda}}(\boldsymbol{t}, \boldsymbol{s}) > u\boldsymbol{b} \right\} \le A\, B\, \mathbb{P}\{\boldsymbol{X}(\boldsymbol{0}) > u\boldsymbol{b}\}, \tag{2.47}$$

where

$$A := \exp\left(-\frac{1}{2} u^2 \boldsymbol{b}^\top \left[\Sigma_u^{-1}(\boldsymbol{\tau}, \boldsymbol{\lambda}) - \Sigma^{-1} \right] \boldsymbol{b} \right), \tag{2.48}$$

$$B := \int_{\mathbb{R}^d} \exp\left(\boldsymbol{b}^\top \Sigma_u^{-1}(\boldsymbol{\tau},\boldsymbol{\lambda})\,\boldsymbol{x}\right) \mathbb{P}\left\{\exists\, \boldsymbol{t},\boldsymbol{s} \in [\boldsymbol{0},\boldsymbol{S}] \colon \boldsymbol{\chi}_{u,\boldsymbol{\tau},\boldsymbol{\lambda}}(\boldsymbol{t},\boldsymbol{s}) > \boldsymbol{x}\right\} d\boldsymbol{x}\,. \tag{2.49}$$

At this point we split the proof in three parts: bounding A, bounding B and comparing the bounds.

Bounding A**.** By (2.107), we have

$$\boldsymbol{b}^\top \left[\Sigma^{-1}(\boldsymbol{\tau},\boldsymbol{\lambda}) - \Sigma^{-1}\right]\boldsymbol{b} \sim \sum_{i=1}^n \left[\boldsymbol{w}^\top A_{2,i}\,\boldsymbol{w}\left[\tau_i^{\beta_i} + \lambda_i^{\beta_i}\right] + \frac{\boldsymbol{w}^\top A_{5,i}\,\boldsymbol{w}}{2}\,|\lambda_i - \tau_i|^{\alpha_i}\right]$$
$$+ \frac{1}{4}\sum_{i,j\in\mathcal{F}} \Xi_{i,j}\left[\tau_i^{\beta_i/2}\,\lambda_j^{\beta_j/2} + \lambda_i^{\beta_i/2}\,\tau_j^{\beta_j/2} + \tau_i^{\beta_i/2}\,\tau_j^{\beta_j/2} + \lambda_i^{\beta_i/2}\,\lambda_j^{\beta_j/2}\right]$$

with the error of order

$$o\left(\sum_{i=1}^n \left[\tau_i^{\beta_i} + \lambda_i^{\beta_i} + |\lambda_i - \tau_i|^{\alpha_i}\right]\right).$$

Using (2.82), we find that

$$\boldsymbol{b}^\top \left[\Sigma^{-1}(\boldsymbol{\tau},\boldsymbol{\lambda}) - \Sigma^{-1}\right]\boldsymbol{b} \ge \frac{1}{2}\sum_{i=1}^n \left[\boldsymbol{w}^\top A_{2,i}\,\boldsymbol{w}\left[\tau_i^{\beta_i} + \lambda_i^{\beta_i}\right] + \frac{\boldsymbol{w}^\top A_{5,i}\,\boldsymbol{w}}{2}\,|\lambda_i - \tau_i|^{\alpha_i}\right]$$

for $\boldsymbol{\tau}$ and $\boldsymbol{\lambda}$ sufficiently close to $\boldsymbol{0}$. Hence,

$$-\frac{1}{2}\,u^2\,\boldsymbol{b}^\top \left[\Sigma_u^{-1}(\boldsymbol{\tau},\boldsymbol{\lambda}) - \Sigma^{-1}\right]\boldsymbol{b} \le -\frac{1}{4}\sum_{i=1}^n \Bigg[u^{2-2\beta_i/\nu_i}\,\boldsymbol{w}^\top A_{2,i}\,\boldsymbol{w}\left[\tau_i^{\beta_i} + \lambda_i^{\beta_i}\right]$$
$$+ u^{2-2\alpha_i/\nu_i}\,\frac{\boldsymbol{w}^\top A_{5,i}\,\boldsymbol{w}}{2}\,|\lambda_i - \tau_i|^{\alpha_i}\Bigg] \tag{2.50}$$

Bounding B**.** As in the proof of Lemma 2.11, our next step consists in deriving a bound for the integral

$$\int_{\mathbb{R}^d} e^{\boldsymbol{b}^\top \Sigma_u^{-1}(\boldsymbol{\tau},\boldsymbol{\lambda})\,\boldsymbol{x}}\,\mathbb{P}\left\{\exists\, \boldsymbol{t},\boldsymbol{s} \in [\boldsymbol{0},\boldsymbol{S}] \colon \boldsymbol{\chi}_{u,\boldsymbol{\tau},\boldsymbol{\lambda}}(\boldsymbol{t},\boldsymbol{s}) > \boldsymbol{x}\right\} d\boldsymbol{x} \le c_1 e^{c_2(G+\sigma^2)}, \tag{2.51}$$

using Lemma 2.10. Here $G \in \mathbb{R}$ and $\sigma^2 > 0$ are any such numbers that for all small enough ε the following two inequalities hold:

$$\begin{aligned} \sup_{F\subset\{1,\dots,d\}}\ \sup_{\boldsymbol{t}\in[\boldsymbol{0},\boldsymbol{\Lambda}]} \boldsymbol{w}_F^\top\,\mathbb{E}\left\{\boldsymbol{\chi}_{u,\boldsymbol{\tau},\boldsymbol{\lambda},F}(\boldsymbol{t},\boldsymbol{s})\right\} &\le G + \varepsilon\sum_{j=1}^d |x_j|, \\ \sup_{F\subset\{1,\dots,d\}}\ \sup_{\boldsymbol{t}\in[\boldsymbol{0},\boldsymbol{\Lambda}]} \mathrm{Var}\left\{\boldsymbol{w}_F^\top\,\boldsymbol{\chi}_{u,\boldsymbol{\tau},\boldsymbol{\lambda},F}(\boldsymbol{t})\right\} &\le \sigma^2. \end{aligned} \tag{2.52}$$

For our current needs they also must be uniform in $\boldsymbol{k}$. The following estimate for $G+\sigma^2$ is proven in the Appendix (see Section 2.6.5):

$$G + \sigma^2 = c_1\sum_{i=1}^n \left[u^{2-2\alpha_i/\nu_i}\left[S_i^{\alpha_i} + S_i\,\left((\lambda_i - \tau_i)\vee S_i\right)^{\alpha_i - 1} + \varepsilon(\lambda_i - \tau_i)^{\alpha_i}\right]\right.$$

$$+ u^{2-2\beta_i/\nu_i} \left[S_i \, (\lambda_i \vee S_i)^{\beta_i - 1} + \varepsilon \, (\lambda_i \vee S_i)^{\beta_i} + \varepsilon^{-1} \, S_j^2 \, (\lambda_j \vee S_j)^{\beta_j - 2} \right] \Bigg]. \quad (2.53)$$

Comparing the bounds. Combining (2.50) for A, the bound (2.51) for B with $G + \sigma^2$ given by (2.53), and plugging all this into the AB bound (2.47), we arrive at the following inequality:

$$\ln \left(\frac{P_{\boldsymbol{b}}(\boldsymbol{\tau}, \boldsymbol{\lambda}, \boldsymbol{S})}{\mathbb{P}\{\boldsymbol{X}(\boldsymbol{0}) > u\boldsymbol{b}\}} \right)$$

$$\leq -\frac{1}{4} \sum_{i=1}^{n} \left[u^{2-2\beta_i/\nu_i} \, \boldsymbol{w}^\top A_{2,i} \, \boldsymbol{w} \left[\tau_i^{\beta_i} + \lambda_i^{\beta_i} \right] + u^{2-2\alpha_i/\nu_i} \, \frac{\boldsymbol{w}^\top A_{5,i} \, \boldsymbol{w}}{4} \, (\lambda_i - \tau_i)^{\alpha_i} \right]$$

$$+ c_1 \sum_{i=1}^{n} \Bigg[u^{2-2\alpha_i/\nu_i} \left[S_i^{\alpha_i} + S_i \, ((\lambda_i - \tau_i) \vee S_i)^{\alpha_i - 1} + \varepsilon (\lambda_i - \tau_i)^{\alpha_i} \right]$$

$$+ u^{2-2\beta_i/\nu_i} \left[S_i \, (\lambda_i \vee S_i)^{\beta_i - 1} + \varepsilon \, (\lambda_i \vee S_i)^{\beta_i} + \varepsilon^{-1} \, S_j^2 \, (\lambda_j \vee S_j)^{\beta_j - 2} \right] \Bigg].$$

Let us exclude the terms

$$\ln H_u(\boldsymbol{\tau}) \coloneqq -\frac{1}{4} \sum_{i=0}^{n} \boldsymbol{w}^\top A_{2,i} \, \boldsymbol{w} \, u^{2-2\beta_i/\nu_i} \, \tau_i^{\beta_i}$$

from further considerations, since it will be useful for us as it is. Note that H_u will appear without alterations in conclusion of the lemma. Setting

$$\varepsilon = \min \left\{ \frac{\boldsymbol{w}^\top A_{5,i} \, \boldsymbol{w}}{32 \, c_1}, \frac{\boldsymbol{w}^\top A_{2,i} \, \boldsymbol{w}}{8 \, c_1} \right\},$$

we find that that the i-th term is at most

$$u^{2-2\beta_i/\nu_i} \left[-\frac{\boldsymbol{w}^\top A_{2,i} \, \boldsymbol{w}}{8} \, \lambda_i^{\beta_i} + c_1 \left[S_i \, (\lambda_i \vee S_i)^{\beta_i - 1} + \varepsilon^{-1} \, S_j^2 \, (\lambda_j \vee S_j)^{\beta_j - 2} \right] \right]$$

$$+ u^{2-2\alpha_i/\nu_i} \left[-\frac{\boldsymbol{w}^\top A_{5,i} \, \boldsymbol{w}}{32} \, (\lambda_i - \tau_i)^{\alpha_i} + c_1 \left[S_i^{\alpha_i} + S_i \, ((\lambda_i - \tau_i) \vee S_i)^{\alpha_i - 1} \right] \right]. \quad (2.54)$$

Case $\tau_i = \lambda_i = 0$. By assumptions of the lemma, this happens if and only if $i \in \mathcal{K} \cup \mathcal{J}$. The right-hand side of (2.54) reads:

$$c_2 \, S_i^{\beta_i} + c_3 \, u^{2-2\alpha_i/\nu_i} \, S_i^{\alpha_i}$$

with some new c_2, $c_3 > 0$. This bound can be further simplified if we note that for $i \in \mathcal{K}$ the second term tends to zero as $u \to \infty$ for a fixed S_i. Hence, this contribution is at most

$$c_2 \, S_i^{\beta_i} + c_4.$$

Case $\lambda_i = \tau_i \neq 0$. By assumptions of the lemma, this happens if and only if $i \in \mathcal{I}' \subsetneq \mathcal{I}$. Note that this set may be empty. The right-hand side of (2.54) reads:

$$u^{2-2\beta_i/\nu_i} \left[-\frac{\boldsymbol{w}^\top A_{2,i} \, \boldsymbol{w}}{4} \, \lambda_i^{\beta_i} + c_1 \left[S_i \, (\lambda_i \vee S_i)^{\beta_i - 1} + \varepsilon^{-1} \, S_j^2 \, (\lambda_j \vee S_j)^{\beta_j - 2} \right] \right] + c_4 \, S_i^{\alpha_i}.$$

There exists N_i such that for all $\lambda_i \geq N_i\, S_i$ this contribution may be bounded from above by

$$-\frac{\boldsymbol{w}^\top A_{2,i}\,\boldsymbol{w}}{8}\, u^{2-2\beta_i/\nu_i}\,\lambda_i^{\beta_i} + c_4\, S_i^{\alpha_i}.$$

Case $\lambda_i - \tau_i > S_i$. By assumptions of the lemma, this condition holds for all $i \in \mathcal{I} \setminus \mathcal{I}'$, and this set is non-empty. The right-hand side of (2.54) reads:

$$u^{2-2\beta_i/\nu_i}\left[-\frac{\boldsymbol{w}^\top A_{2,i}\,\boldsymbol{w}}{4}\,\lambda_i^{\beta_i} + c_1\Big[S_i\,\lambda_i^{\beta_i-1} + \varepsilon^{-1}\,S_j^2\,\lambda_j^{\beta_j-2}\Big]\right]$$
$$+\left[-\frac{\boldsymbol{w}^\top A_{5,i}\,\boldsymbol{w}}{32}\,(\lambda_i-\tau_i)^{\alpha_i} + c_5 S_i\,(\lambda_i-\tau_i)^{\alpha_i-1}\right].$$

There exists N_i such that for all $\lambda_i - \tau_i \geq N_i S_i$ this contribution is at most

$$-\frac{\boldsymbol{w}^\top A_{2,i}\,\boldsymbol{w}}{8}\, u^{2-2\beta_i/\nu_i}\,\lambda_i^{\beta_i} - \frac{\boldsymbol{w}^\top A_{5,i}\,\boldsymbol{w}}{64}\,(\lambda_i-\tau_i)^{\alpha_i}.$$

Combined bound. We have obtained the following inequality: if

$$\lambda_i \geq N_i\,S_i \quad \text{for} \quad i \in \mathcal{I}', \quad \text{and} \quad \lambda_i - \tau_i \geq N_i\,S_i \quad \text{for} \quad i \in \mathcal{I}\setminus\mathcal{I}', \tag{2.55}$$

then

$$\ln\left(\frac{P_{\boldsymbol{b}}(\boldsymbol{\tau},\boldsymbol{\lambda},\boldsymbol{S})}{c_7\,H_u(\boldsymbol{\tau})\,\mathbb{P}\{\boldsymbol{X}(\boldsymbol{0}) > u\boldsymbol{b}\}}\right) \leq -\frac{1}{8}\sum_{i\in\mathcal{I}} u^{2-2\beta_i/\nu_i}\,\boldsymbol{w}^\top A_{2,i}\,\boldsymbol{w}\,\lambda_i^{\beta_i}$$
$$-\frac{1}{64}\sum_{i\in\mathcal{I}\setminus\mathcal{I}'}\boldsymbol{w}^\top A_{5,i}\,\boldsymbol{w}\,(\lambda_i-\tau_i)^{\alpha_i} + c_6\sum_{i\in\mathcal{I}'} S_i^{\alpha_i} + c_6\sum_{i\in\mathcal{J}\cup\mathcal{K}} S_i^{\beta_i}. \tag{2.56}$$

Next, we want to lift conditions (2.55).

Lifting the condition $\lambda_i - \tau_i \geq N_i\,S_i$. Assume that there is exactly one $i \in \mathcal{I}\setminus\mathcal{I}'$ such that $\lambda_i - \tau_i < N_i\,S_i$. The general case can be addressed in the same way. Define $\Delta := (\lambda_i - \tau_i - S_i)/N_i$, and slice $P_{\boldsymbol{b}}$ in the i-th direction using this new scale:

$$P_{\boldsymbol{b}}(\boldsymbol{\tau},\boldsymbol{\lambda},\boldsymbol{S}) \leq \sum_{j=\tau_i/\Delta}^{(\tau_i+S_i)/\Delta}\;\sum_{k=\lambda_i/\Delta}^{(\lambda_i+S_i)/\Delta} P_{\boldsymbol{b}}(k,j,\Delta) \leq S_i^2\,\Delta^{-2}\,P_{\boldsymbol{b}}\left(j^*\Delta,\, k_*\,\Delta,\, \Delta\right), \tag{2.57}$$

where $j^* = (\tau_i + S_i)/\Delta$, $k_* = \lambda_i/\Delta$, and

$$P_{\boldsymbol{b}}(\tau,\lambda,\Delta) = \mathbb{P}\left\{\begin{aligned} &\exists\, t_i \in u^{-2/\nu_i}[\tau,\tau+\Delta], \\ &\exists\, \boldsymbol{t}' \in u^{-2/\boldsymbol{\nu}'}[\boldsymbol{\tau}',\boldsymbol{\tau}'+\boldsymbol{S}']: \quad \boldsymbol{X}(\boldsymbol{t}) > u\boldsymbol{b}, \\ &\exists\, s_i \in u^{-2/\nu_i}[\lambda,\lambda+\Delta], \\ &\exists\, \boldsymbol{s}' \in u^{-2/\boldsymbol{\nu}'}[\boldsymbol{\lambda}',\boldsymbol{\lambda}'+\boldsymbol{S}']: \quad \boldsymbol{X}(\boldsymbol{s}) > u\boldsymbol{b} \end{aligned}\right\}, \qquad \boldsymbol{x}' = (x_j)_{j\neq i}.$$

Note that

$$k_*\,\Delta - j^*\Delta = \lambda_i - \tau_i - S_i = N_i\Delta,$$

which means that we can apply (2.56). Therefore,

$$\ln\left(\frac{P_{\boldsymbol{b}}(\boldsymbol{\tau},\boldsymbol{\lambda},\boldsymbol{S})}{c_7\,H_u(\boldsymbol{\tau})\,\mathbb{P}\{\boldsymbol{X}(\boldsymbol{0})>u\boldsymbol{b}\}\prod_{i\in\mathcal{I}\setminus\mathcal{I}'}S_i^2\,\Delta^{-2}}\right)\le-\frac{1}{8}\sum_{i\in\mathcal{I}}u^{2-2\beta_i/\nu_i}\,\boldsymbol{w}^\top A_{2,i}\,\boldsymbol{w}\,\lambda_i^{\beta_i}$$

$$-\frac{1}{64}\sum_{i\in\mathcal{I}\setminus\mathcal{I}'}\boldsymbol{w}^\top A_{5,i}\,\boldsymbol{w}\,(\lambda_i-\tau_i-S_i)^{\alpha_i}+c_6\sum_{i\in\mathcal{I}'}S_i^{\alpha_i}+c_6\sum_{i\in\mathcal{J}\cup\mathcal{K}}S_i^{\beta_i}. \quad (2.58)$$

which is now valid without the second condition of (2.55).

Lifting the condition $\lambda_i\ge N_i\,S_i$. Assume now that there is exactly one $i\in\mathcal{I}'$ such that $\lambda_i<N_i\,S_i$. Recall that for $i\in\mathcal{I}'$ holds $\lambda_i=\tau_i$. Using the same approach, take $\Delta\coloneqq(\lambda_i\wedge1)/N_i$. Then,

$$P_{\boldsymbol{b}}(\boldsymbol{\tau},\boldsymbol{\lambda},\boldsymbol{S})\le\sum_{k\neq j}P_{\boldsymbol{b}}(k\Delta,j\Delta,\Delta)+\sum_{k=j}P_{\boldsymbol{b}}(j\Delta,j\Delta,\Delta). \quad (2.59)$$

where the sums are taken over

$$\left\{(k,j)\in\mathbb{Z}_+^2\colon\lambda_i/\Delta\le k,\,j\le(\lambda_i+S_i)/\Delta\right\}.$$

Note that all the results up to this point were valid without any assumptions on $\boldsymbol{S}>\boldsymbol{0}$. Now, assuming that S_i is large enough, we can use the bounds proven above to show that

$$\sum_{|k-j|\ge2}P_{\boldsymbol{b}}(k\Delta,j\Delta,\Delta)\le c_8\sum_{k=j}P_{\boldsymbol{b}}(j\Delta,j\Delta,\Delta).$$

It can also be shown that

$$\sum_{|k-j|=1}P_{\boldsymbol{b}}(k\Delta,j\Delta,\Delta)\le c_9\sum_{k=j}P_{\boldsymbol{b}}(j\Delta,j\Delta,\Delta).$$

It is therefore enough to bound the second sum of (2.59). We have

$$P_{\boldsymbol{b}}(\boldsymbol{\tau},\boldsymbol{\lambda},\boldsymbol{S})\le c_9\,S_i\,\Delta^{-1}\,P_{\boldsymbol{b}}(j_*\Delta,j_*\Delta,\Delta). \quad (2.60)$$

Since $j_*\Delta=\lambda_i\ge\lambda_i\wedge1=N_i\,\Delta$, the bound (2.58) is applicable. The i-th term of the right-hand side is

$$-\frac{\boldsymbol{w}^\top A_{2,i}\,\boldsymbol{w}}{8}\,u^{2-2\beta_i/\nu_i}\,(j_*\,\Delta)^{\beta_i}+c_6\,\Delta_i^{\alpha_i}.$$

Note that $j_*=\lambda_i/\Delta$, so the first term did not change, whereas the second is now bounded by a constant: $c_6(\lambda_i\wedge1)^{\alpha_i}\le c_6$. We have thus shown that the following bound

$$\ln\left(\frac{P_{\boldsymbol{b}}(\boldsymbol{\tau},\boldsymbol{\lambda},\boldsymbol{S})}{c_{10}\,H_u(\boldsymbol{\tau})\,\mathbb{P}\{\boldsymbol{X}(\boldsymbol{0})>u\boldsymbol{b}\}\prod_{i\in\mathcal{I}\setminus\mathcal{I}'}S_i^2\,\Delta^{-2}\prod_{i\in\mathcal{I}}S_i\,\Delta^{-1}}\right)$$

$$\le-\frac{1}{8}\sum_{i\in\mathcal{I}}u^{2-2\beta_i/\nu_i}\,\boldsymbol{w}^\top A_{2,i}\,\boldsymbol{w}\,\lambda_i^{\beta_i}-\frac{1}{64}\sum_{i\in\mathcal{I}\setminus\mathcal{I}'}\boldsymbol{w}^\top A_{5,i}\,\boldsymbol{w}\,(\lambda_i-\tau_i-S_i)^{\alpha_i}+c_6\sum_{i\in\mathcal{J}\cup\mathcal{K}}S_i^{\beta_i} \quad (2.61)$$

is valid without conditions (2.55). This concludes the proof. □

2.5.3 Positivity of constant

Lemma 2.13. *For all $\boldsymbol{t}\in\mathbb{R}^n$ holds*

$$\int_{\mathbb{R}^d}e^{\boldsymbol{1}^\top\boldsymbol{x}}\,\mathbb{P}\left\{\boldsymbol{Y}_{\boldsymbol{\nu},\mathbb{V}}(\boldsymbol{t})+\boldsymbol{Z}(\boldsymbol{t})-\boldsymbol{d}_{\boldsymbol{\nu},\mathbb{W}}(\boldsymbol{t})>\boldsymbol{x}\right\}d\boldsymbol{x}=\exp\left(-\sum_{i\in\mathcal{J}\cup\mathcal{K}}|t_i|^{\nu_i}\boldsymbol{1}^\top W_i\,\boldsymbol{1}+\boldsymbol{1}^\top R_{\boldsymbol{Z}}(\boldsymbol{t},\boldsymbol{t})\,\boldsymbol{1}\right).$$

Proof. Set

$$\widetilde{\boldsymbol{Y}}_{\boldsymbol{\nu},\mathbb{V}}(\boldsymbol{t}) = \sum_{i\in\mathcal{I}\cup\mathcal{J}} \boldsymbol{Y}_{\nu_i,V_i}(t_i), \qquad \widetilde{\boldsymbol{d}}_{\boldsymbol{\nu},\mathbb{W}}(\boldsymbol{t}) = \sum_{i\in\mathcal{I}\cup\mathcal{J}} S_{\nu_i,V_i}(t_i)\boldsymbol{1} + \sum_{j\in\mathcal{J}\cup\mathcal{K}} |t_j|^{\nu_i} W_i \boldsymbol{1} \tag{2.62}$$

and note that

$$\boldsymbol{Y}_{\boldsymbol{\nu},\mathbb{V}}(\boldsymbol{t}) - \boldsymbol{d}_{\boldsymbol{\nu},\mathbb{W}}(\boldsymbol{t}) = \widetilde{\boldsymbol{Y}}_{\boldsymbol{\nu},\mathbb{V}}(\boldsymbol{t}) - \widetilde{\boldsymbol{d}}_{\boldsymbol{\nu},\mathbb{W}}(\boldsymbol{t}).$$

The claim follows from

$$\begin{aligned}
&\int_{\mathbb{R}^d} e^{\boldsymbol{1}^\top \boldsymbol{x}} \mathbb{P}\left\{\boldsymbol{Y}_{\boldsymbol{\nu},\mathbb{V}}(\boldsymbol{t}) + \boldsymbol{Z}(\boldsymbol{t}) - \boldsymbol{d}_{\boldsymbol{\nu},\mathbb{W}}(\boldsymbol{t}) > \boldsymbol{x}\right\} d\boldsymbol{x} \\
&\quad = \mathbb{E}\left\{\int_{\mathbb{R}^d} e^{\boldsymbol{1}^\top \boldsymbol{x}} \mathbb{1}\left\{\boldsymbol{Y}_{\boldsymbol{\nu},\mathbb{V}}(\boldsymbol{t}) + \boldsymbol{Z}(\boldsymbol{t}) - \boldsymbol{d}_{\boldsymbol{\nu},\mathbb{W}}(\boldsymbol{t}) > \boldsymbol{x}\right\} d\boldsymbol{x}\right\} \\
&\quad = \exp\left(-\boldsymbol{1}^\top \widetilde{\boldsymbol{d}}_{\boldsymbol{\nu},\mathbb{W}}(\boldsymbol{t})\right) \mathbb{E}\left\{\exp\left(\boldsymbol{1}^\top \widetilde{\boldsymbol{Y}}_{\boldsymbol{\nu},\mathbb{V}}(\boldsymbol{t})\right)\right\} \mathbb{E}\left\{\exp\left(\boldsymbol{1}^\top \boldsymbol{Z}(\boldsymbol{t})\right)\right\} \\
&\quad = \exp\left(-\sum_{i\in\mathcal{J}\cup\mathcal{K}} |t_i|^{\nu_i} \boldsymbol{1}^\top W_i \boldsymbol{1}\right) \exp\left(-\sum_{i\in\mathcal{I}\cup\mathcal{J}} \boldsymbol{1}^\top S_{\nu_i,V_i}(t_i)\boldsymbol{1}\right) \\
&\quad \times \exp\left(\frac{1}{2}\sum_{i\in\mathcal{I}\cup\mathcal{J}} \boldsymbol{1}^\top R_{\nu_i,V_i}(t_i,t_i)\boldsymbol{1}\right) \exp\left(\boldsymbol{1}^\top R_{\boldsymbol{Z}}(\boldsymbol{t},\boldsymbol{t})\boldsymbol{1}\right)
\end{aligned}$$

along with $R_{\alpha,V}(t,t) = 2S_{\alpha,V}(t)$. □

Lemma 2.14. *For all* $\boldsymbol{t},\ \boldsymbol{s} \in \mathbb{R}^n$ *holds*

$$\begin{aligned}
&\int_{\mathbb{R}^d} e^{\boldsymbol{1}^\top \boldsymbol{x}}\, \mathbb{P}\left\{\begin{matrix}\boldsymbol{Y}_{\boldsymbol{\nu},\mathbb{V}}(\boldsymbol{t}) + \boldsymbol{Z}(\boldsymbol{t}) - \boldsymbol{d}_{\boldsymbol{\nu},\mathbb{W}}(\boldsymbol{t}) > \boldsymbol{x} \\ \boldsymbol{Y}_{\boldsymbol{\nu},\mathbb{V}}(\boldsymbol{s}) + \boldsymbol{Z}(\boldsymbol{s}) - \boldsymbol{d}_{\boldsymbol{\nu},\mathbb{W}}(\boldsymbol{s}) > \boldsymbol{x}\end{matrix}\right\} d\boldsymbol{x} \\
&\quad \le \exp\left(-\frac{1}{2}\sum_{i\in\mathcal{J}\cup\mathcal{K}} \left[|t_i|^{\nu_i} + |s_i|^{\nu_i}\right] \boldsymbol{1}^\top W_i \boldsymbol{1}\right) \exp\left(-\frac{1}{4}\sum_{i\in\mathcal{I}\cup\mathcal{J}} \boldsymbol{1}^\top S_{\nu_i,V_i}(t_i - s_i)\boldsymbol{1}\right) \\
&\qquad \times \exp\left(\frac{1}{2}\boldsymbol{1}^\top R_{\boldsymbol{Z}}(\boldsymbol{t},\boldsymbol{t}) + R_{\boldsymbol{Z}}(\boldsymbol{t},\boldsymbol{s}) + R_{\boldsymbol{Z}}(\boldsymbol{s},\boldsymbol{t}) + R_{\boldsymbol{Z}}(\boldsymbol{s},\boldsymbol{s})\boldsymbol{1}\right).
\end{aligned}$$

Proof. Set $\widetilde{\boldsymbol{Y}}_{\boldsymbol{\nu},\mathbb{V}}$ and $\widetilde{\boldsymbol{d}}_{\boldsymbol{\nu},\mathbb{W}}$ as in (2.62). By

$$\mathbb{P}\left\{\begin{matrix}\boldsymbol{Y}_{\boldsymbol{\nu},\mathbb{V}}(\boldsymbol{t}) + \boldsymbol{Z}(\boldsymbol{t}) - \boldsymbol{d}_{\boldsymbol{\nu},\mathbb{W}}(\boldsymbol{t}) > \boldsymbol{x} \\ \boldsymbol{Y}_{\boldsymbol{\nu},\mathbb{V}}(\boldsymbol{s}) + \boldsymbol{Z}(\boldsymbol{s}) - \boldsymbol{d}_{\boldsymbol{\nu},\mathbb{W}}(\boldsymbol{s}) > \boldsymbol{x}\end{matrix}\right\} \le \mathbb{P}\left\{\begin{matrix}\boldsymbol{Y}_{\boldsymbol{\nu},\mathbb{V}}(\boldsymbol{t}) - \boldsymbol{d}_{\boldsymbol{\nu},\mathbb{W}}(\boldsymbol{t}) + \boldsymbol{Z}(\boldsymbol{t}) \\ \quad + \boldsymbol{Y}_{\boldsymbol{\nu},\mathbb{V}}(\boldsymbol{s}) - \boldsymbol{d}_{\boldsymbol{\nu},\mathbb{W}}(\boldsymbol{s}) + \boldsymbol{Z}(\boldsymbol{s}) > 2\boldsymbol{x}\end{matrix}\right\},$$

we have

$$\begin{aligned}
&\int_{\mathbb{R}^d} e^{\boldsymbol{1}^\top \boldsymbol{x}}\, \mathbb{P}\left\{\begin{matrix}\boldsymbol{Y}_{\boldsymbol{\nu},\mathbb{V}}(\boldsymbol{t}) + \boldsymbol{Z}(\boldsymbol{t}) - \boldsymbol{d}_{\boldsymbol{\nu},\mathbb{W}}(\boldsymbol{t}) > \boldsymbol{x} \\ \boldsymbol{Y}_{\boldsymbol{\nu},\mathbb{V}}(\boldsymbol{s}) + \boldsymbol{Z}(\boldsymbol{s}) - \boldsymbol{d}_{\boldsymbol{\nu},\mathbb{W}}(\boldsymbol{s}) > \boldsymbol{x}\end{matrix}\right\} d\boldsymbol{x} \le \exp\left(-\frac{1}{2}\boldsymbol{1}^\top\left[\widetilde{\boldsymbol{d}}_{\boldsymbol{\nu},\mathbb{W}}(\boldsymbol{t}) + \widetilde{\boldsymbol{d}}_{\boldsymbol{\nu},\mathbb{W}}(\boldsymbol{s})\right]\right) \\
&\quad \times \mathbb{E}\left\{\exp\left(\frac{1}{2}\boldsymbol{1}^\top\left[\widetilde{\boldsymbol{Y}}_{\boldsymbol{\nu},\mathbb{V}}(\boldsymbol{t}) + \widetilde{\boldsymbol{Y}}_{\boldsymbol{\nu},\mathbb{V}}(\boldsymbol{s})\right]\right)\right\} \mathbb{E}\left\{\exp\left(\frac{1}{2}\boldsymbol{1}^\top\left(\boldsymbol{Z}(\boldsymbol{t}) + \boldsymbol{Z}(\boldsymbol{s})\right)\boldsymbol{\mathcal{N}}\right)\right\}.
\end{aligned}$$

First, compute the last expectation:

$$\mathbb{E}\left\{\exp\left(\frac{1}{2}\boldsymbol{1}^\top\left(\boldsymbol{Z}(\boldsymbol{t}) + \boldsymbol{Z}(\boldsymbol{s})\right)\right)\right\} = \exp\left(\frac{1}{2}\boldsymbol{1}^\top\left[R_{\boldsymbol{Z}}(\boldsymbol{t},\boldsymbol{t}) + R_{\boldsymbol{Z}}(\boldsymbol{t},\boldsymbol{s}) + R_{\boldsymbol{Z}}(\boldsymbol{s},\boldsymbol{t}) + R_{\boldsymbol{Z}}(\boldsymbol{s},\boldsymbol{s})\right]\boldsymbol{1}\right).$$

Since

$$\begin{aligned}
&\mathbb{E}\left\{\exp\left(\frac{1}{2}\mathbf{1}^\top\left[\widetilde{\boldsymbol{Y}}_{\boldsymbol{\nu},\mathbb{V}}(\boldsymbol{t})+\widetilde{\boldsymbol{Y}}_{\boldsymbol{\nu},\mathbb{V}}(\boldsymbol{s})\right]\right)\right\}\\
&\quad=\exp\left(\frac{1}{8}\mathbf{1}^\top\mathbb{E}\left\{\left[\widetilde{\boldsymbol{Y}}_{\boldsymbol{\nu},\mathbb{V}}(\boldsymbol{t})+\widetilde{\boldsymbol{Y}}_{\boldsymbol{\nu},\mathbb{V}}(\boldsymbol{s})\right]\left[\widetilde{\boldsymbol{Y}}_{\boldsymbol{\nu},\mathbb{V}}(\boldsymbol{t})+\widetilde{\boldsymbol{Y}}_{\boldsymbol{\nu},\mathbb{V}}(\boldsymbol{s})\right]^\top\right\}\mathbf{1}\right)\\
&\quad=\exp\left(\frac{1}{8}\sum_{i\in\mathcal{I}\cup\mathcal{J}}\mathbf{1}^\top\left[R_{\nu_i,V_i}(t_i,t_i)+R_{\nu_i,V_i}(t_i,s_i)+R_{\nu_i,V_i}(s_i,t_i)+R_{\nu_i,V_i}(s_i,s_i)\right]\mathbf{1}\right)\\
&\quad=\exp\left(\frac{1}{4}\sum_{i\in\mathcal{I}\cup\mathcal{J}}\mathbf{1}^\top\left[S_{\nu_i,V_i}(t_i)+R_{\nu_i,V_i}(t_i,s_i)+S_{\nu_i,V_i}(s_i)\right]\mathbf{1}\right)
\end{aligned}$$

and

$$\mathbf{1}^\top\left[\widetilde{\boldsymbol{d}}_{\boldsymbol{\nu},\mathbb{W}}(\boldsymbol{t})+\widetilde{\boldsymbol{d}}_{\boldsymbol{\nu},\mathbb{W}}(\boldsymbol{s})\right]=\sum_{i\in\mathcal{I}\cup\mathcal{J}}\mathbf{1}^\top\left[S_{\nu_i,V_i}(t_i)+S_{\nu_i,V_i}(s_i)\right]\mathbf{1}+\sum_{i\in\mathcal{J}\cup\mathcal{K}}\left[|t_i|^{\nu_i}+|t_i|^{\nu_i}\right]\mathbf{1}^\top W_i\,\mathbf{1},$$

we have

$$\begin{aligned}
&\exp\left(-\frac{1}{2}\mathbf{1}^\top\left[\widetilde{\boldsymbol{d}}_{\boldsymbol{\nu},\mathbb{W}}(\boldsymbol{t})+\widetilde{\boldsymbol{d}}_{\boldsymbol{\nu},\mathbb{W}}(\boldsymbol{s})\right]\right)\mathbb{E}\left\{\exp\left(\frac{1}{2}\mathbf{1}^\top\left[\widetilde{\boldsymbol{Y}}_{\boldsymbol{\nu},\mathbb{V}}(\boldsymbol{t})+\widetilde{\boldsymbol{Y}}_{\boldsymbol{\nu},\mathbb{V}}(\boldsymbol{s})\right]\right)\right\}\\
&\quad=\exp\left(-\frac{1}{2}\sum_{i\in\mathcal{J}\cup\mathcal{K}}\left[|t_i|^{\nu_i}+|s_i|^{\nu_i}\right]\mathbf{1}^\top W_i\mathbf{1}\right)\\
&\quad\quad\times\exp\left(-\frac{1}{4}\sum_{i\in\mathcal{I}\cup\mathcal{J}}\mathbf{1}^\top\left[S_{\nu_i,V_i}(t_i)+S_{\nu_i,V_i}(s_i)-R_{\nu_i,V_i}(t_i,s_i)\right]\mathbf{1}\right)
\end{aligned}$$

and the claim follows by $S_{\alpha,V}(t)+S_{\alpha,V}(s)-R_{\alpha,V}(t,s)=S_{\alpha,V}(t-s)$. □

Lemma 2.15 (Lower bound for the constant)**.** *For $\boldsymbol{S}_{\mathcal{J}\cup\mathcal{K}}/2<\boldsymbol{\delta}_{\mathcal{J}\cup\mathcal{K}}<\boldsymbol{S}_{\mathcal{J}\cup\mathcal{K}}$ and all $\boldsymbol{\delta}_{\mathcal{I}}>\mathbf{0}$, holds*

$$H_{\boldsymbol{\nu},\mathbb{V},\mathbb{W}}\left([\mathbf{0},\boldsymbol{S}]\right)\prod_{i\in\mathcal{I}}\frac{1}{S_i}\geq\prod_{i\in\mathcal{I}}\frac{1}{\delta_i}\left[1-\sum_{\mathcal{I}_0\subsetneq\mathcal{I}}\prod_{i\in\mathcal{I}\setminus\mathcal{I}_0}\frac{A_i}{\delta_i}\right]\tag{2.63}$$

with

$$A_i=2\left(\frac{4}{\mathbf{1}^\top V_i\mathbf{1}}\right)^{1/\nu_i}\Gamma\left(\frac{1}{\nu_i}+1\right)$$

if $\mathbf{1}^\top V_i\,\mathbf{1}>0$ for all $i\in\mathcal{I}$.

Proof. For any $\boldsymbol{\delta}>\mathbf{0}$ we have

$$\begin{aligned}
H_{\boldsymbol{\nu},\mathbb{V},\mathbb{W}}\left([\mathbf{0},\boldsymbol{S}]\right)&\geq\int_{\mathbb{R}^d}e^{\mathbf{1}^\top\boldsymbol{x}}\,\mathbb{P}\left\{\exists\,\boldsymbol{t}\in[\mathbf{0},\boldsymbol{S}]\cap\boldsymbol{\delta}\mathbb{Z}_+^n:\boldsymbol{Y}_{\boldsymbol{\nu},\mathbb{V}}(\boldsymbol{t})+\boldsymbol{Z}(\boldsymbol{t})-\boldsymbol{d}_{\boldsymbol{\nu},\mathbb{W}}(\boldsymbol{t})>\boldsymbol{x}\right\}d\boldsymbol{x}\\
&\geq\sum_{\boldsymbol{k}\leq\boldsymbol{N_\delta}}\int_{\mathbb{R}^d}e^{\mathbf{1}^\top\boldsymbol{x}}\,\mathbb{P}\left\{\boldsymbol{Y}_{\boldsymbol{\nu},\mathbb{V}}(\boldsymbol{\delta k})+\boldsymbol{Z}(\boldsymbol{\delta k})-\boldsymbol{d}_{\boldsymbol{\nu},\mathbb{W}}(\boldsymbol{\delta k})>\boldsymbol{x}\right\}d\boldsymbol{x}\\
&\quad-\sum_{\substack{\boldsymbol{k},\boldsymbol{l}\leq\boldsymbol{N_\delta},\\\boldsymbol{k}\neq\boldsymbol{l}}}\int_{\mathbb{R}^d}e^{\mathbf{1}^\top\boldsymbol{x}}\,\mathbb{P}\left\{\begin{matrix}\boldsymbol{Y}_{\boldsymbol{\nu},\mathbb{V}}(\boldsymbol{\delta k})+\boldsymbol{Z}(\boldsymbol{\delta k})-\boldsymbol{d}_{\boldsymbol{\nu},\mathbb{W}}(\boldsymbol{\delta k})>\boldsymbol{x}\\\boldsymbol{Y}_{\boldsymbol{\nu},\mathbb{V}}(\boldsymbol{\delta l})+\boldsymbol{Z}(\boldsymbol{\delta l})-\boldsymbol{d}_{\boldsymbol{\nu},\mathbb{W}}(\boldsymbol{\delta l})>\boldsymbol{x}\end{matrix}\right\}d\boldsymbol{x},
\end{aligned}$$

where $\boldsymbol{N_\delta} = \lfloor \boldsymbol{S}/\boldsymbol{\delta} \rfloor$. Take $\boldsymbol{S}_{\mathcal{J}\cup\mathcal{K}}/2 < \boldsymbol{\delta}_{\mathcal{J}\cup\mathcal{K}} < \boldsymbol{S}_{\mathcal{J}\cup\mathcal{K}}$. Then $\boldsymbol{N}_{\boldsymbol{\delta},\mathcal{J}\cup\mathcal{K}} = \mathbf{1}_{\mathcal{J}\cup\mathcal{K}}$, and therefore $\boldsymbol{k}_{\mathcal{I}\cup\mathcal{K}} = \mathbf{0}_{\mathcal{J}\cup\mathcal{K}}$. It follows by definition of $\boldsymbol{Z}$ that $\boldsymbol{Z}(\boldsymbol{\delta k}) = \mathbf{0}$. Apply Lemma 2.13 and Lemma 2.14.

$$\begin{aligned} H_{\boldsymbol{\nu},\mathbb{V},\mathbb{W}}([\mathbf{0},\boldsymbol{S}]) &\geq \sum_{\boldsymbol{k}_\mathcal{I} \leq \boldsymbol{N}_{\boldsymbol{\delta},\mathcal{I}}} 1 - \sum_{\substack{\boldsymbol{k}_\mathcal{I}, \boldsymbol{l}_\mathcal{I} \leq \boldsymbol{N}_{\boldsymbol{\delta},\mathcal{I}}, \\ \boldsymbol{k}_\mathcal{I} \neq \boldsymbol{l}_\mathcal{I}}} \exp\left(-\frac{1}{4}\sum_{i\in\mathcal{I}} \mathbf{1}^\top S_{\nu_i,V_i}(\delta_i k_i - \delta_i l_i)\mathbf{1}\right) \\ &= \prod_{i\in\mathcal{I}} \frac{S_i}{\delta_i} - \Sigma_2(\boldsymbol{\delta}). \end{aligned} \tag{2.64}$$

To bound the double sum, use $S_{\alpha,V} = |t|^\alpha \left(V 1_{t\geq 0} + V^\top 1_{t<0}\right)$ reindex the sum as follows: let $\mathcal{I}_0(\boldsymbol{k})$ denote those indices, in which $k_i = l_i$. This set cannot be equal to the entire $\mathcal{I}$, becasuse in this case $\boldsymbol{k}_\mathcal{I} = \boldsymbol{l}_\mathcal{I}$ and such pairs are excluded from the sum, but it is empty if $\min_{i\in\mathcal{I}} |k_i - l_i| \geq 1$.

$$\begin{aligned} \Sigma_2(\boldsymbol{\delta}) &= \sum_{\mathcal{I}_0 \subsetneq \mathcal{I}} \prod_{i\in\mathcal{I}\setminus\mathcal{I}_0} \sum_{k=1}^{N_{\boldsymbol{\delta},i}} \sum_{l=k+1}^{N_{\boldsymbol{\delta},i}} 2\exp\left(-\frac{1}{4}\sum_{i\in\mathcal{I}} \delta_i^{\nu_i} |k_i - l_i|^{\nu_i} \mathbf{1}^\top V_i \mathbf{1}\right) \prod_{i\in\mathcal{I}_0} \sum_{k=1}^{N_{\boldsymbol{\delta},i}} 1 \\ &\leq \sum_{\mathcal{I}_0 \subsetneq \mathcal{I}} \prod_{i\in\mathcal{I}\setminus\mathcal{I}_0} \frac{2S_i}{\delta_i^2} \left(\frac{4}{\mathbf{1}^\top V_i \mathbf{1}}\right)^{1/\nu_i} \int_0^\infty e^{-x^{\nu_i}}\, dx \prod_{i\in\mathcal{I}_0} \sum_{k=1}^{N_{\boldsymbol{\delta},i}} 1 \\ &= \sum_{\mathcal{I}_0 \subsetneq \mathcal{I}} \prod_{i\in\mathcal{I}\setminus\mathcal{I}_0} \frac{2S_i}{\delta_i^2} \left(\frac{4}{\mathbf{1}^\top V_i \mathbf{1}}\right)^{1/\nu_i} \Gamma\left(\frac{1}{\nu_i}+1\right) \prod_{i\in\mathcal{I}_0} \frac{S_i}{\delta_i} \\ &= \prod_{j\in\mathcal{I}} S_i \sum_{\mathcal{I}_0 \subsetneq \mathcal{I}} \prod_{i\in\mathcal{I}\setminus\mathcal{I}_0} \frac{2}{\delta_i^2} \left(\frac{4}{\mathbf{1}^\top V_i \mathbf{1}}\right)^{1/\nu_i} \Gamma\left(\frac{1}{\nu_i}+1\right) \prod_{i\in\mathcal{I}_0} \frac{1}{\delta_i} \end{aligned}$$

Combining the above together, we obtain

$$H_{\boldsymbol{\nu},\mathbb{V},\mathbb{W}}([\mathbf{0},\boldsymbol{S}]) \prod_{i\in\mathcal{I}} \frac{1}{S_i} \geq \prod_{i\in\mathcal{I}} \frac{1}{\delta_i} - \sum_{\mathcal{I}_0 \subsetneq \mathcal{I}} \prod_{i\in\mathcal{I}\setminus\mathcal{I}_0} \frac{A_i}{\delta_i^2} \prod_{i\in\mathcal{I}_0} \frac{1}{\delta_i}, \tag{2.65}$$

where

$$A_i = 2\left(\frac{4}{\mathbf{1}^\top V_i \mathbf{1}}\right)^{1/\nu_i} \Gamma\left(\frac{1}{\nu_i}+1\right).$$

This concludes the proof. □

2.5.4 Proof of Theorem 2.1

Proof. By Lemma 2.18, the generalized variance satisfies

$$\sigma_{\boldsymbol{b}}^{-2}(\boldsymbol{\tau}) - \sigma_{\boldsymbol{b}}^{-2}(\mathbf{0}) = \sum_{i,j=1}^n \Xi_{i,j}\, \tau_i^{\beta_i/2} \tau_j^{\beta_j/2} + o\left(\sum_{i=1}^n \tau_i^{\beta_i}\right),$$

with Ξ defined in (2.81). By (2.82) and A2.4, there exists a constant $c > 0$ such that

$$\sigma_{\boldsymbol{b}}^{-2}(\boldsymbol{\tau}) - \sigma_{\boldsymbol{b}}^{-2}(\mathbf{0}) \geq c\sum_{i=1}^n \tau_i^{\beta_i},$$

and therefore we obtain

$$\mathbb{P}\left\{\exists\, \boldsymbol{t} \in [\mathbf{0},\boldsymbol{T}] \colon \boldsymbol{X}(\boldsymbol{t}) > u\boldsymbol{b}\right\} \sim \mathbb{P}\left\{\exists\, \boldsymbol{t} \in [\mathbf{0},\boldsymbol{\delta}_u] \colon \boldsymbol{X}(\boldsymbol{t}) > u\boldsymbol{b}\right\}$$

with $\boldsymbol{\delta}_u = u^{-2/\boldsymbol{\beta}} \ln^{2/\boldsymbol{\beta}} u$ by using the Piterbarg inequality (2.25).

Upper bound

Take $\mathbf{\Lambda} > \mathbf{0}$ and use the log-layer bound (Lemma 2.11) to obtain

$$\begin{aligned}\mathbb{P}\left\{\exists \boldsymbol{t} \in [\mathbf{0}, \boldsymbol{\delta}_u] \colon \boldsymbol{X}(\boldsymbol{t}) > u\boldsymbol{b}\right\} &\le \mathbb{P}\left\{\exists \boldsymbol{t} \in [\mathbf{0}, u^{-2/\boldsymbol{\beta}}\mathbf{\Lambda}] \colon \boldsymbol{X}(\boldsymbol{t}) > u\boldsymbol{b}\right\} \\ &\quad + C\,\mathbb{P}\left\{\boldsymbol{X}(\mathbf{0}) > u\boldsymbol{b}\right\} \exp\left(-\frac{1}{8}\sum_{i=1}^{n} \xi_i\, \Lambda_i^{\beta_i}\right). \end{aligned} \tag{2.66}$$

In order to bound the first term from above, let us split the cube $u^{-2/\boldsymbol{\beta}}[\mathbf{0}, \mathbf{\Lambda}]$ into parts of "size" $u^{-2/\boldsymbol{\nu}}[\mathbf{0}, \boldsymbol{S}]$ with $\boldsymbol{S} > \mathbf{0}$ such that $\boldsymbol{S}_{\mathcal{I}^c} = \mathbf{\Lambda}_{\mathcal{I}^c}$, and note that

$$\begin{aligned}\mathbb{P}\left\{\exists \boldsymbol{t} \in u^{-2/\boldsymbol{\nu}}[\mathbf{0}, \mathbf{\Lambda}] \colon \boldsymbol{X}(\boldsymbol{t}) > u\boldsymbol{b}\right\} &\le \sum_{\boldsymbol{k} \le \boldsymbol{N}_u} \mathbb{P}\left\{\exists \boldsymbol{t} \in u^{-2/\boldsymbol{\nu}}\boldsymbol{S}[\boldsymbol{k}, \boldsymbol{k}+1] \colon \boldsymbol{X}(\boldsymbol{t}) > u\boldsymbol{b}\right\} \\ &=: \Sigma_1(u, \mathbf{\Lambda}, \boldsymbol{S}),\end{aligned} \tag{2.67}$$

where $\boldsymbol{N}_u = \left\lceil u^{2/\boldsymbol{\nu} - 2/\boldsymbol{\beta}}\mathbf{\Lambda}/\boldsymbol{S}\right\rceil$. By Lemma 2.21 and the uniform local Pickands Lemma 2.9, we have

$$\begin{aligned}&\mathbb{P}\left\{\exists \boldsymbol{t} \in u^{-2/\boldsymbol{\nu}}\boldsymbol{S}\left[\boldsymbol{k}, \boldsymbol{k}+\mathbf{1}\right] : \boldsymbol{X}(\boldsymbol{t}) > u\boldsymbol{b}\right\} \\ &\qquad \sim H_{\boldsymbol{\nu}, \mathbb{V}_{\boldsymbol{w}}, \mathbb{W}_{\boldsymbol{w}}}\left([\mathbf{0}, \boldsymbol{S}]\right) \mathbb{P}\left\{\boldsymbol{X}_{u,\boldsymbol{k}}(\mathbf{0}) > u\boldsymbol{b}\right\} \\ &\qquad \sim H_{\boldsymbol{\nu}, \mathbb{V}_{\boldsymbol{w}}, \mathbb{W}_{\boldsymbol{w}}}\left([\mathbf{0}, \boldsymbol{S}]\right) \mathbb{P}\left\{\boldsymbol{X}(\mathbf{0}) > u\boldsymbol{b}\right\} \exp\left(-\frac{u^2}{2}\boldsymbol{b}^\top\left[\Sigma_{u,\boldsymbol{k}}^{-1} - \Sigma^{-1}\right]\boldsymbol{b}\right),\end{aligned}$$

and therefore by (2.79)

$$\begin{aligned}\mathbb{P}\left\{\exists \boldsymbol{t} \in u^{-2/\boldsymbol{\nu}}\boldsymbol{S}\left[\boldsymbol{k}, \boldsymbol{k}+\mathbf{1}\right] : \boldsymbol{X}(\boldsymbol{t}) > u\boldsymbol{b}\right\} &\sim H_{\boldsymbol{\nu}, \mathbb{V}_{\boldsymbol{w}}, \mathbb{W}_{\boldsymbol{w}}}\left([\mathbf{0}, \boldsymbol{S}]\right) \mathbb{P}\left\{\boldsymbol{X}(\mathbf{0}) > u\boldsymbol{b}\right\} \\ &\quad \times \exp\left(-\frac{1}{2}\sum_{i,j \in \mathcal{I}} \Xi_{i,j}\left(S_i\, k_i\, u^{-\zeta_i}\right)^{\beta_i/2}\left(S_j\, k_j\, u^{-\zeta_j}\right)^{\beta_j/2}\right)\end{aligned}$$

with Ξ defined in (2.81) and $\boldsymbol{\zeta} = 2/\boldsymbol{\beta} - 2/\boldsymbol{\nu}$. Using the following formula

$$\sum_{\boldsymbol{k} \le \boldsymbol{N}_u} \exp\left(-\frac{1}{2}\sum_{i,j \in \mathcal{I}} \Xi_{i,j}\left(S_i\, k_i\, u^{-\zeta_i}\right)^{\beta_i/2}\left(S_j\, k_j\, u^{-\zeta_j}\right)^{\beta_j/2}\right) \sim \prod_{i \in \mathcal{I}} \frac{u^{\zeta_i}}{S_i}\, G(\boldsymbol{\beta}, \Xi, \mathbf{\Lambda}),$$

where

$$G(\boldsymbol{\beta}, \Xi, \mathbf{\Lambda}) \coloneqq \int_{\mathbf{0}_{\mathcal{I}}}^{\mathbf{\Lambda}_{\mathcal{I}}} \exp\left(-\frac{1}{2}\sum_{i,j \in \mathcal{I}} \Xi_{i,j}\, t_i^{\beta_i/2}\, t_j^{\beta_j}\right) d\boldsymbol{t}, \quad G(\boldsymbol{\beta}, \Xi) \coloneqq \lim_{\mathbf{\Lambda} \to \infty} G(\boldsymbol{\beta}, \Xi) < \infty,$$

which may be proven by Riemann sum argument, we obtain the following estimate for the single sum:

$$\frac{\Sigma_1(u, \mathbf{\Lambda}, \boldsymbol{S})}{\mathbb{P}\left\{\boldsymbol{X}(\mathbf{0}) > u\boldsymbol{b}\right\}} \sim H_{\boldsymbol{\nu}, \mathbb{V}_{\boldsymbol{w}}, \mathbb{V}_{\boldsymbol{w}}}\left([\mathbf{0}, \boldsymbol{S}]\right) \prod_{i \in \mathcal{I}} \frac{u^{\zeta_i}}{S_i}\, G(\boldsymbol{\beta}, \Xi, \mathbf{\Lambda}).$$

Let us prove that the following limit

$$\mathcal{H}_{\boldsymbol{\nu},\mathbb{V}_{\boldsymbol{w}},\mathbb{W}_{\boldsymbol{w}}} = \lim_{\boldsymbol{\Lambda}_{\mathcal{I}^c}\to\infty} \lim_{\boldsymbol{S}_{\mathcal{I}}\to\infty} H_{\boldsymbol{\nu},\mathbb{V}_{\boldsymbol{w}},\mathbb{W}_{\boldsymbol{w}}}\left([\mathbf{0},\boldsymbol{S}]\right) \prod_{i\in\mathcal{I}} \frac{1}{S_i}, \tag{2.68}$$

exists and is finite. Since $\boldsymbol{S} \mapsto H_{\boldsymbol{\nu},\mathbb{V}_{\boldsymbol{w}},\mathbb{W}_{\boldsymbol{w}}}\left([\mathbf{0},\boldsymbol{S}]\right)$ is subadditive in each S_i, the limit

$$\lim_{\boldsymbol{S}_{\mathcal{I}}\to\infty} H_{\boldsymbol{\nu},\mathbb{V}_{\boldsymbol{w}},\mathbb{W}_{\boldsymbol{w}}}\left([\mathbf{0},\boldsymbol{S}]\right) \prod_{i\in\mathcal{I}} \frac{1}{S_i}$$

exists and is an increasing function of each $S_i > 0$, $i \in \mathcal{I}^c$. To prove that the limit (2.68) exists and is finite, it would suffice to show that $H_{\boldsymbol{\nu},\mathbb{V}_{\boldsymbol{w}},\mathbb{W}_{\boldsymbol{w}}}\left([\mathbf{0},\boldsymbol{S}]\right) / \prod_{i\in\mathcal{I}} S_i$ is uniformly bounded in $\boldsymbol{S}_{\mathcal{I}^c}$. To this end, fix $\boldsymbol{\Lambda}_0 < \boldsymbol{\Lambda}$ and note that for every $\boldsymbol{S}$ and $\boldsymbol{S}_0$ such that $\boldsymbol{S}_{\mathcal{I}^c} = \boldsymbol{\Lambda}_{\mathcal{I}^c}$ and $\boldsymbol{S}_{0,\mathcal{I}^c} = \boldsymbol{\Lambda}_{0,\mathcal{I}}$, holds

$$\begin{aligned}
\Sigma_1'(u,\boldsymbol{\Lambda},\boldsymbol{S}) &\le \mathbb{P}\left\{\exists\, \boldsymbol{t} \in u^{-2/\boldsymbol{\nu}}[\mathbf{0},\boldsymbol{\Lambda}] \colon \boldsymbol{X}(\boldsymbol{t}) > u\boldsymbol{b}\right\} \\
&\le \mathbb{P}\left\{\exists\, \boldsymbol{t} \in u^{-2/\boldsymbol{\nu}}[\mathbf{0},\boldsymbol{\Lambda}_0] \colon \boldsymbol{X}(\boldsymbol{t}) > u\boldsymbol{b}\right\} + \mathbb{P}\left\{\exists\, \boldsymbol{t} \in u^{-2/\boldsymbol{\nu}}[\boldsymbol{\Lambda}_0,\boldsymbol{\Lambda}] \colon \boldsymbol{X}(\boldsymbol{t}) > u\boldsymbol{b}\right\} \\
&\le \Sigma_1(u,\boldsymbol{\Lambda}_0,\boldsymbol{S}_0) + C\,\mathbb{P}\left\{\boldsymbol{X}(\mathbf{0}) > u\boldsymbol{b}\right\} \exp\left(-\frac{1}{8}\sum_{i=1}^n \xi_i\, \Lambda_{0,i}^{\beta_i}\right),
\end{aligned}$$

where $\Sigma_1'(u,\boldsymbol{\Lambda},\boldsymbol{S})$ is the same single sum (2.67), but with $\boldsymbol{N}_u - 1$ instead of $\boldsymbol{N}_u$ in the limit of summation. It is easy to see that a computation analogous to what we did above for $\Sigma_1(u,\boldsymbol{\Lambda},\boldsymbol{S})$ gives the same estimate for $\Sigma_1'(u,\boldsymbol{\Lambda},\boldsymbol{S})$:

$$\frac{\Sigma_1'(u,\boldsymbol{\Lambda},\boldsymbol{S})}{\mathbb{P}\left\{\boldsymbol{X}(\mathbf{0}) > u\boldsymbol{b}\right\}} \sim H_{\boldsymbol{\nu},\mathbb{V}_{\boldsymbol{w}},\mathbb{W}_{\boldsymbol{w}}}\left([\mathbf{0},\boldsymbol{S}]\right) \prod_{i\in\mathcal{I}} \frac{u^{\zeta_i}}{S_i}\, G(\boldsymbol{\beta},\boldsymbol{\Xi}). \tag{2.69}$$

Hence,

$$\begin{aligned}
H_{\boldsymbol{\nu},\mathbb{V}_{\boldsymbol{w}},\mathbb{W}_{\boldsymbol{w}}}\left([\mathbf{0},\boldsymbol{S}]\right) \prod_{i\in\mathcal{I}} \frac{1}{S_i}\, G(\boldsymbol{\beta},\Xi,\boldsymbol{\Lambda}) &= \lim_{u\to\infty} \frac{\Sigma_1'(u,\boldsymbol{\Lambda},\boldsymbol{S})}{\mathbb{P}\left\{\boldsymbol{X}(\mathbf{0}) > u\boldsymbol{b}\right\}\prod_{i\in\mathcal{I}} u^{\zeta_i}} \\
&\le \lim_{u\to\infty} \frac{\Sigma_1(u,\boldsymbol{\Lambda}_0,\boldsymbol{S}_0)}{\mathbb{P}\left\{\boldsymbol{X}(\mathbf{0}) > u\boldsymbol{b}\right\}\prod_{i\in\mathcal{I}} u^{\zeta_i}} + C\exp\left(-\frac{1}{8}\sum_{i=1}^n \xi_i\, \Lambda_i^{\beta_i}\right) 1_{\mathcal{I}=\varnothing} \\
&\le H_{\boldsymbol{\nu},\mathbb{V}_{\boldsymbol{w}},\mathbb{W}_{\boldsymbol{w}}}\left([\mathbf{0},\boldsymbol{S}_0]\right) \prod_{i\in\mathcal{I}} \frac{1}{S_{0,i}}\, G(\boldsymbol{\beta},\Xi,\boldsymbol{\Lambda}_0) + C\exp\left(-\frac{1}{8}\sum_{i=1}^n \xi_i\, \Lambda_{0,i}^{\beta_i}\right) 1_{\mathcal{I}=\varnothing} =: c,
\end{aligned}$$

Since for $\Lambda_i > 1$ holds $G(\boldsymbol{\beta},\boldsymbol{\Xi},\boldsymbol{\Lambda}) > G(\boldsymbol{\beta},\Xi,\mathbf{1}) = c_1 > 0$, we have the uniform bound

$$H_{\boldsymbol{\nu},\mathbb{V}_{\boldsymbol{w}},\mathbb{W}_{\boldsymbol{w}}}\left([\mathbf{0},\boldsymbol{S}]\right) \prod_{i\in\mathcal{I}} \frac{1}{S_i} \le \frac{c}{c_1},$$

and thus the claim is proved. Using (2.66), letting $\boldsymbol{S} \to \infty$ and then $\boldsymbol{\Lambda} \to \infty$, we obtain

$$\limsup_{u\to\infty} \frac{\mathbb{P}\left\{\exists\, \boldsymbol{t} \in u^{-2/\beta}[\mathbf{0},\boldsymbol{\Lambda}] \colon \boldsymbol{X}(\boldsymbol{t}) > u\boldsymbol{b}\right\}}{\mathbb{P}\left\{\boldsymbol{X}(\mathbf{0}) > u\boldsymbol{b}\right\}\prod_{i\in\mathcal{I}} u^{\zeta_i}} \le \mathcal{H}_{\boldsymbol{\nu},\mathbb{V}_{\boldsymbol{w}},\mathbb{W}_{\boldsymbol{w}}}\, G(\boldsymbol{\beta},\Xi).$$

Lower bound

By Bonferroni inequality,

$$\mathbb{P}\left\{\exists \boldsymbol{t} \in u^{-2/\boldsymbol{\nu}}[\boldsymbol{0},\boldsymbol{\Lambda}]\colon \boldsymbol{X}(\boldsymbol{t}) > u\boldsymbol{b}\right\} \ge \Sigma_1'(u,\boldsymbol{\Lambda},\boldsymbol{S}) - \Sigma_2(u,\boldsymbol{\Lambda},\boldsymbol{S}), \tag{2.70}$$

where $\Sigma_1'(u,\boldsymbol{\Lambda},\boldsymbol{S})$ is the same single sum (2.67), but with $\boldsymbol{N}_u - \boldsymbol{1}$ instead of $\boldsymbol{N}_u$ in the limit of summation, and

$$\Sigma_2(u,\boldsymbol{\Lambda},\boldsymbol{S}) = \sum_{\boldsymbol{k}\neq\boldsymbol{l}\le\boldsymbol{N}_u} P_{\boldsymbol{b}}(\boldsymbol{S}\boldsymbol{k},\boldsymbol{S}\boldsymbol{l},\boldsymbol{S})$$

and it only remains to bound $\Sigma_2(u,\boldsymbol{\Lambda},\boldsymbol{S})$. Note that if $\boldsymbol{k}\le\boldsymbol{N}_u$, then in particular $\boldsymbol{k}_{\mathcal{I}^c} = \boldsymbol{0}$.

First, rewrite it as $\Sigma_2 = \Sigma_{2,1} + \Sigma_{2,2}$, where the double sums $\Sigma_{2,1}$ and $\Sigma_{2,2}$ are taken over

$$\left\{\begin{matrix}\boldsymbol{k}_\mathcal{I}\neq\boldsymbol{l}_\mathcal{I}\le\boldsymbol{N}_{u,\mathcal{I}},\\ \min_{i\in\mathcal{I}}|k_i-l_i|\le 1\\ \boldsymbol{k}_{\mathcal{I}^c}=\boldsymbol{l}_{\mathcal{I}^c}=\boldsymbol{0}\end{matrix}\right\} \quad\text{and}\quad \left\{\begin{matrix}\boldsymbol{k}_\mathcal{I}\neq\boldsymbol{l}_\mathcal{I}\le\boldsymbol{N}_{u,\mathcal{I}},\\ \min_{i\in\mathcal{I}}|k_i-l_i|> 1\\ \boldsymbol{k}_{\mathcal{I}^c}=\boldsymbol{l}_{\mathcal{I}^c}=\boldsymbol{0}\end{matrix}\right\}$$

correspondingly. Each term of the sum $\Sigma_{2,2}(u,\boldsymbol{\Lambda},\boldsymbol{S})$ satisfies conditions of Lemma 2.12, and therefore

$$\begin{aligned}&\frac{\Sigma_{2,2}(u,\boldsymbol{\Lambda},\boldsymbol{S})}{H_u(\boldsymbol{S}\boldsymbol{k})\,\mathbb{P}\left\{\boldsymbol{X}(\boldsymbol{0})>u\boldsymbol{b}\right\}}\\ &\le C\,e^{c\sum_{j\in\mathcal{I}^c}\Lambda_j^{\beta_j}} \sum_{\boldsymbol{k}_\mathcal{I}\le\boldsymbol{N}_{u,\mathcal{I}}}\ \sum_{\boldsymbol{k}_\mathcal{I}+\boldsymbol{1}_\mathcal{I}\le\boldsymbol{l}_\mathcal{I}\le\boldsymbol{N}_{u,\mathcal{I}}}\ \prod_{i\in\mathcal{I}}(l_i-k_i-1)^2\exp\left(-\frac{\varkappa_i}{64}S_i^{\alpha_i}(l_i-k_i-1)^{\alpha_i}\right)\\ &\le Ce^{c\sum_{j\in\mathcal{I}^c}\Lambda_j^{\beta_j}}\prod_{i\in\mathcal{I}}\sum_{l=1}^{\infty} l^2\exp\left(-\frac{\varkappa}{64}S_i^{\alpha_i}\,l^{\alpha_i}\right).\end{aligned}$$

It remains to note that

$$\sum_{k=0}^{N_{u,j}}\exp\left(-\frac{\xi_i}{4}S_j^{\beta_j}k^{\beta_j}u^{-\beta_j\zeta_j}\right)\sim\frac{4^{1/\beta_i}u^{\zeta_i}}{S_i\,\xi_i^{1/\beta_i}}\int_0^{\Lambda_i}e^{-x^{\beta_i}}\,dx, \tag{2.71}$$

which together with (2.69) gives

$$\frac{\Sigma_{2,2}(u,\boldsymbol{\Lambda},\boldsymbol{S})}{\Sigma_1'(u,\boldsymbol{\Lambda},\boldsymbol{S})}\le\frac{C\,e^{c\sum_{j\in\mathcal{I}^c}\Lambda_j^{\beta_j}}}{H_{\boldsymbol{\nu},\mathbb{V}_{\boldsymbol{w}},\mathbb{W}_{\boldsymbol{w}}}\left([\boldsymbol{0},\boldsymbol{S}]\right)}\prod_{i\in\mathcal{I}}\exp\left(-\frac{\varkappa_i}{64}S_i^{\alpha_i}\right)$$

and therefore

$$\lim_{\boldsymbol{\Lambda}_{\mathcal{I}^c}\to\infty}\ \lim_{\boldsymbol{S}_\mathcal{I}\to\infty}\ \limsup_{u\to\infty}\frac{\Sigma_{2,2}(u,\boldsymbol{\Lambda},\boldsymbol{S})}{\Sigma_1'(u,\boldsymbol{\Lambda},\boldsymbol{S})}=0.$$

Next, we bound the sum $\Sigma_{2,1}$ over adjacent events, that is, those with $\min_{i\in\mathcal{I}}|k_i-l_i|\le 1$. To this end, introduce the following reindexing of the sum: for a given pair $(\boldsymbol{k},\boldsymbol{l})$, indexing $\Sigma_{2,1}$, let $\mathcal{I}_0(\boldsymbol{k},\boldsymbol{l})$ denote those indices, in which $\boldsymbol{l}_{\mathcal{I}_0}=\boldsymbol{k}_{\mathcal{I}_0}$. This set may be empty, but it cannot be equal to $\mathcal{I}$, since in this case $\boldsymbol{l}_\mathcal{I}$ would be equal to $\boldsymbol{k}_\mathcal{I}$. Let also $\mathcal{I}_1(\boldsymbol{k},\boldsymbol{l})$ denote those indices, in which $\boldsymbol{l}_{\mathcal{I}_1}=\boldsymbol{k}_{\mathcal{I}_1}+1$. This set may be equal to $\mathcal{I}$, but it cannot be empty, since these terms are already included in $\Sigma_{2,2}(u,\boldsymbol{\Lambda},\boldsymbol{S})$. Finally, denote and $\mathcal{I}_2=\mathcal{I}\setminus(\mathcal{I}_0\cup\mathcal{I}_1)$. We have

$$\Sigma_{2,1}(u,\boldsymbol{\Lambda},\boldsymbol{S})=\sum_{\mathcal{I}_0\subsetneq\mathcal{I}}\ \sum_{\varnothing\neq\mathcal{I}_1\subset\mathcal{I}}\ \sum_{\boldsymbol{k}_{\mathcal{I}_0}=\boldsymbol{l}_{\mathcal{I}_0}\le\boldsymbol{N}_{u,\mathcal{I}_0}}\ \sum_{\substack{\boldsymbol{l}_{\mathcal{I}_1}=\boldsymbol{k}_{\mathcal{I}_1}+1,\\ \boldsymbol{k}_{\mathcal{I}_1}\le\boldsymbol{N}_{u,\mathcal{I}_1}}}\ \sum_{\substack{\boldsymbol{k}_{\mathcal{I}_2}\neq\boldsymbol{l}_{\mathcal{I}_2}\le\boldsymbol{N}_{u,\mathcal{I}_2},\\ \min_{i\in\mathcal{I}_2}|k_i-l_i|>1}} 2P_{\boldsymbol{b}}(\boldsymbol{S}\boldsymbol{k},\boldsymbol{S}\boldsymbol{l},\boldsymbol{S}).$$

Let us split $[\mathbf{0}, \boldsymbol{S}]$ along axes from $\mathcal{I}_1$ into $2^{|\mathcal{I}_1|}$ subintervals

$$[\mathbf{0}, \boldsymbol{S}] = \boldsymbol{S}\left[\mathbf{1}_{\mathcal{I}_1}\boldsymbol{S}^{-1/2}, \mathbf{1}\right] \cup \bigcup_{j=0}^{2^{|\mathcal{I}_1|}-1} C_j, \qquad \mathbf{1}_{\mathcal{I}_1,i} = \begin{cases} 1, & i \in \mathcal{I}_1, \\ 0, & i \notin \mathcal{I}_1, \end{cases}$$

and use it to obtain the following bound:

$$P_{\boldsymbol{b}}(\boldsymbol{Sk}, \boldsymbol{Sl}, \boldsymbol{S}) \leq A(\boldsymbol{k}, \boldsymbol{l}) + \sum_{j=0}^{2^{|\mathcal{I}_1|}-1} \mathbb{P}\left\{\exists\, \boldsymbol{t} \in C_j \colon \boldsymbol{X}(\boldsymbol{t}) > u\boldsymbol{b}\right\} =: \Sigma'_{2,1}(u, \boldsymbol{\Lambda}, \boldsymbol{S}) + \Sigma''_{2,1}(u, \boldsymbol{\Lambda}, \boldsymbol{S}). \tag{2.72}$$

where

$$A(\boldsymbol{k}, \boldsymbol{l}) = \mathbb{P}\left\{\begin{array}{ll} \exists\, \boldsymbol{t} \in u^{-2/\boldsymbol{\nu}}\boldsymbol{S}\left[\boldsymbol{k}, \boldsymbol{k}+\mathbf{1}\right] : & \boldsymbol{X}(\boldsymbol{t}) > u\boldsymbol{b} \\ \exists\, \boldsymbol{s}_{\mathcal{I}_0} \in u^{-2/\boldsymbol{\nu}_{\mathcal{I}_0}}\boldsymbol{S}_{\mathcal{I}_0}\left[\boldsymbol{k}_{\mathcal{I}_0}, \boldsymbol{k}_{\mathcal{I}_0} + \mathbf{1}_{\mathcal{I}_0}\right], & \\ \exists\, \boldsymbol{s}_{\mathcal{I}_1} : u^{-2/\boldsymbol{\nu}_{\mathcal{I}_1}}\boldsymbol{S}_{\mathcal{I}_1}\left[\boldsymbol{S}_{\mathcal{I}_1}^{-1/2} + \boldsymbol{k}_{\mathcal{I}_1} + \mathbf{1}, \boldsymbol{k}_{\mathcal{I}_1} + \mathbf{2}_{\mathcal{I}_1}\right], & \\ \exists\, \boldsymbol{s}_{\mathcal{I}_2} \in u^{-2/\boldsymbol{\nu}_{\mathcal{I}_2}}\boldsymbol{S}_{\mathcal{I}_2}\left[\boldsymbol{l}_{\mathcal{I}_2}, \boldsymbol{l}_{\mathcal{I}_2} + \mathbf{1}_{\mathcal{I}_2}\right] : & \boldsymbol{X}(\boldsymbol{s}) > u\boldsymbol{b} \end{array}\right\}$$

In order to apply Lemma 2.12, we lengthen the $\mathcal{I}_1$ interval interval in the definition of $A(\boldsymbol{k}, \boldsymbol{l})$ by $\boldsymbol{S}_{\mathcal{I}'}^{1/2}$ so that it fell under the definition of double event probabilty (2.45), and therefore

$$\begin{aligned} \frac{A(\boldsymbol{k}, \boldsymbol{l})}{\mathbb{P}\left\{\boldsymbol{X}(\mathbf{0}) > u\boldsymbol{b}\right\}} &\leq \frac{P_{\boldsymbol{b}}(\boldsymbol{Sk}, \boldsymbol{Sk}\mathbf{1}_{\mathcal{I}_0\cup\mathcal{I}_1} + \boldsymbol{S}^{1/2}\mathbf{1}_{\mathcal{I}_1} + \mathbf{1}_{\mathcal{I}_1} + \boldsymbol{Sl}\mathbf{1}_{\mathcal{I}_2}, \boldsymbol{S})}{\mathbb{P}\left\{\boldsymbol{X}(\mathbf{0}) > u\boldsymbol{b}\right\}} \\ &\leq C \prod_{i_c \in \mathcal{I}^c} A_{c,i_c} \prod_{i_0 \in \mathcal{I}_0} A_{0,i_0} \prod_{i_1 \in \mathcal{I}_1} A_{1,i_1} \prod_{i_2 \in \mathcal{I}_2} A_{2,i_2}, \end{aligned}$$

where $A_{c,i} = \exp\left(c\Lambda_i^{\alpha_i}\right)$, $A_{0,i} = S_i$

$$A_{1,i} = S_i \exp\left(-\frac{\xi_i(\boldsymbol{w})}{32} S_i^{\alpha_i/2} - \frac{\tau_i(\boldsymbol{w})}{4} S_i^{\beta_i} k_i^{\beta_i} u^{2-2\beta_i/\nu_i}\right),$$

$$A_{2,i} = (l_i - k_i - 1)^{-2} \exp\left(-\frac{\xi_i(\boldsymbol{w})}{32}(l_i - k_i - 1)^{\alpha_i} - \frac{\tau_i(\boldsymbol{w})}{4} S_i^{\beta_i} k_i^{\beta_i} u^{2-2\beta_i/\nu_i}\right).$$

Summing up $\Sigma'_{2,1}(u, \boldsymbol{\Lambda}, \boldsymbol{S})$ in $\boldsymbol{k}$ and $\boldsymbol{l}$, we obtain

$$\frac{\Sigma'_{2,1}(u, \boldsymbol{\Lambda}, \boldsymbol{S})}{\mathbb{P}\left\{\boldsymbol{X}(\mathbf{0}) > u\boldsymbol{b}\right\}} \leq C \prod_{i_c \in \mathcal{I}^c} \mathbb{A}_{c,i_c} \sum_{\mathcal{I}_0 \subsetneq \mathcal{I}} \sum_{\varnothing \neq \mathcal{I}_1 \subset \mathcal{I}} \prod_{i_0 \in \mathcal{I}_0} \mathbb{A}_{0,i_0} \prod_{i_1 \in \mathcal{I}_1} \mathbb{A}_{1,i_1} \prod_{i_2 \in \mathcal{I}_2} \mathbb{A}_{2,i_2},$$

where $\mathbb{A}_{c,i} = A_{c,i} = \exp\left(c\Lambda_i^{\alpha_i}\right)$, $\mathbb{A}_{0,i} = S_i u^{\zeta_i}\Lambda_i / S_i = u^{\zeta_i}\Lambda_i$ and

$$\mathbb{A}_{1,i} = \frac{u^{\zeta_i}}{S_i}\left(\frac{4}{\tau_i(\boldsymbol{w})}\right)^{1/\beta_i} \Gamma\left(\frac{1}{\beta_i} + 1\right) S_i \exp\left(-\frac{\xi_i(\boldsymbol{w})}{32} S_i^{\alpha_i/2}\right),$$

$$\mathbb{A}_{2,i} = \frac{u^{\zeta_i}}{S_i}\left(\frac{4}{\tau_i(\boldsymbol{w})}\right)^{1/\beta_i} \Gamma\left(\frac{1}{\beta_i} + 1\right) \exp\left(-\frac{\xi_i(\boldsymbol{w})}{32} S_i^{\alpha_i}\right).$$

Therefore, by (2.69), (2.71) and Lemma 2.15 to bound $1/H_{\boldsymbol{\nu}, \mathbb{V}_{\boldsymbol{w}}, \mathbb{W}_{\boldsymbol{w}}}$ from above by $C / \prod_{i \in \mathcal{I}} S_i$, we obtain

$$\frac{\Sigma'_{2,1}(u,\boldsymbol{\Lambda},\boldsymbol{S})}{\Sigma'_1(u,\boldsymbol{\Lambda},\boldsymbol{S})} \le C \prod_{i_c\in\mathcal{I}^c} \exp\left(c\Lambda_{i_c}^{\alpha_{i_c}}\right) \times$$

$$\times \sum_{\mathcal{I}_0\subsetneq\mathcal{I}} \sum_{\varnothing\neq\mathcal{I}_1\subset\mathcal{I}} \prod_{i_0\in\mathcal{I}_0} \frac{\Lambda_{i_0}}{S_i} \prod_{i_1\in\mathcal{I}_1} \exp\left(-\frac{\xi_{i_1}(\boldsymbol{w})}{32} S_i^{\alpha_{i_1}/2}\right) \prod_{i_2\in\mathcal{I}_2} \frac{1}{S_i} \exp\left(-\frac{\xi_{i_2}(\boldsymbol{w})}{32} S_{i_2}^{\alpha_{i_2}}\right)$$

and

$$\lim_{\boldsymbol{\Lambda}_{\mathcal{I}^c}\to\infty} \lim_{\boldsymbol{S}_{\mathcal{I}}\to\infty} \limsup_{u\to\infty} \frac{\Sigma'_{2,1}(u,\boldsymbol{\Lambda},\boldsymbol{S})}{\Sigma'_1(u,\boldsymbol{\Lambda},\boldsymbol{S})} = 0.$$

Finally, let us find a bound for the second term of (2.72). By local Pickands lemma

$$\mathbb{P}\left\{\exists \boldsymbol{t}\in C_j\colon \boldsymbol{X}(\boldsymbol{t}) > u\boldsymbol{b}\right\} \sim H_{\boldsymbol{\nu},\mathbb{V}_{\boldsymbol{w}},\mathbb{W}_{\boldsymbol{w}}}\left(\left[\boldsymbol{0}, \boldsymbol{1}_{\mathcal{I}_1}\boldsymbol{S}^{1/2} + \boldsymbol{1}_{\mathcal{I}\setminus\mathcal{I}_1}\boldsymbol{S}\right]\right) \mathbb{P}\left\{\boldsymbol{X}(\boldsymbol{0}) > u\boldsymbol{b}\right\}$$

and therefore, using $N_{u,i} = \lceil \Lambda_i u^{\zeta_i}/S_i \rceil$, obtain

$$\Sigma''_{2,1}(u,\boldsymbol{\Lambda},\boldsymbol{S}) \coloneqq \sum_{\boldsymbol{k}\le\boldsymbol{N}_u} \sum_{\varnothing\neq\mathcal{I}_1\subset\mathcal{I}} \sum_{j=0}^{2^{|\mathcal{I}_1|}-1} \mathbb{P}\left\{\exists \boldsymbol{t}\in C_j\colon \boldsymbol{X}(\boldsymbol{t}) > u\boldsymbol{b}\right\}$$

$$\sim \mathbb{P}\left\{\boldsymbol{X}(\boldsymbol{0}) > u\boldsymbol{b}\right\} \prod_{i\in\mathcal{I}} \frac{\Lambda_i u^{\zeta_i}}{S_i} \sum_{\varnothing\neq\mathcal{I}_1\subset\mathcal{I}} 2^{|\mathcal{I}_1|} H_{\boldsymbol{\nu},\mathbb{V}_{\boldsymbol{w}},\mathbb{W}_{\boldsymbol{w}}}\left(\left[\boldsymbol{0}, \boldsymbol{1}_{\mathcal{I}_1}\boldsymbol{S}^{1/2} + \boldsymbol{1}_{\mathcal{I}\setminus\mathcal{I}_1}\boldsymbol{S}\right]\right)$$

and

$$\frac{\Sigma''_{2,1}(u,\boldsymbol{\Lambda},\boldsymbol{S})}{\Sigma'_1(u,\boldsymbol{\Lambda},\boldsymbol{S})} \le \left[\prod_{i\in\mathcal{I}} \Lambda_i\right] \sum_{\varnothing\neq\mathcal{I}_1\subset\mathcal{I}} 2^{|\mathcal{I}_1|} H_{\boldsymbol{\nu},\mathbb{V}_{\boldsymbol{w}},\mathbb{W}_{\boldsymbol{w}}}\left(\left[\boldsymbol{0}, \boldsymbol{1}_{\mathcal{I}_1}\boldsymbol{S}^{1/2} + \boldsymbol{1}_{\mathcal{I}\setminus\mathcal{I}_1}\boldsymbol{S}\right]\right) \prod_{j\in\mathcal{I}} \frac{1}{S_i}.$$

By subadditivity of the generalized Pickands-Piterbarg constant,

$$\lim_{\boldsymbol{S}\to\infty} H_{\boldsymbol{\nu},\mathbb{V}_{\boldsymbol{w}},\mathbb{W}_{\boldsymbol{w}}}\left(\left[\boldsymbol{0}, \boldsymbol{1}_{\mathcal{I}'}\boldsymbol{S}^{1/2} + \boldsymbol{1}_{\mathcal{I}\setminus\mathcal{I}'}\boldsymbol{S}\right]\right) \prod_{j\in\mathcal{I}} \frac{1}{S_i} \le \lim_{\boldsymbol{S}\to\boldsymbol{\infty}} H_{\boldsymbol{\nu},\mathbb{V}_{\boldsymbol{w}},\mathbb{W}_{\boldsymbol{w}}}\left([\boldsymbol{0},\boldsymbol{1}]\right) \prod_{j\in\mathcal{I}_1} S_j^{-1/2} = 0,$$

and therefore

$$\lim_{\boldsymbol{\Lambda}_{\mathcal{I}^c}\to\infty} \lim_{\boldsymbol{S}_{\mathcal{I}}\to\infty} \lim_{u\to\infty} \frac{\Sigma''_{2,1}(u,\boldsymbol{\Lambda},\boldsymbol{S})}{\Sigma'_1(u,\boldsymbol{\Lambda},\boldsymbol{S})} = 0.$$

□

2.6 Appendix

2.6.1 Covariance lemma

Lemma 2.16. *Let $D = (D_{i,j})_{i,j\in\mathcal{F}}$ be a block matrix, each block of which is a $d\times d$ matrix. Then the following are equivalent:*

1. *$D \succcurlyeq 0$.*
2. *There exists a family $(C_{i,k})_{i,k\in\mathcal{F}}$ of $d\times d$ matrices such that*

$$D_{i,j} = \sum_{k\in\mathcal{F}} C_{i,k}\, C_{k,j}^\top.$$

3. A matrix-valued function R defined by

$$R(\boldsymbol{t},\boldsymbol{s}) = \sum_{i,j\in\mathcal{F}} D_{i,j}\, t_i\, s_j, \qquad \boldsymbol{t},\boldsymbol{s}\in\mathbb{R}^n$$

is a covariance function of some $\mathbb{R}^d$-Gaussian random field.

If either of statements hold, then R is the covariance function of

$$\boldsymbol{Z}(\boldsymbol{t}) \coloneqq \sum_{k\in\mathcal{F}} C_k(\boldsymbol{t})\,\boldsymbol{\mathcal{N}}_k, \quad C_k(\boldsymbol{t}) \coloneqq \sum_{i\in\mathcal{F}} C_{i,k}\, t_i,$$

where $\boldsymbol{\mathcal{N}}_k \sim N(\boldsymbol{0}, I)$ are independent standard Gaussian vectors.

Proof of Lemma 2.16. Let $n = |\mathcal{F}|$. If $D \succcurlyeq 0$, then there exists a $(nd)\times(nd)$ matrix C such that $D = CC^\top$. By viewing C as a block matrix with the same block structure as D, we find that

$$D_{i,j} = \sum_{k\in\mathcal{F}} C_{i,k}\, C_{k,j}^\top.$$

Hence,

$$R(\boldsymbol{t},\boldsymbol{s}) = \sum_{i,j\in\mathcal{F}}\sum_{k\in\mathcal{F}} C_{i,k}\, C_{k,j}^\top\, t_i\, s_j = \sum_{k\in\mathcal{F}} \left(\sum_{i\in\mathcal{F}} C_{i,k}\, t_i\right)\left(\sum_{j\in\mathcal{F}} C_{j,k}\, s_j\right)^\top,$$

which clearly is positive definite.

If, on the other hand, R is positive definite, then for all collections $\{(\boldsymbol{t}_i,\boldsymbol{z}_i)\in\mathbb{R}^n\times\mathbb{R}^{\mathcal{F}},\ i = 1,\ldots,p\}$ holds

$$0 \le \sum_{k,l=1}^p \boldsymbol{z}_k^\top R(\boldsymbol{t}_k,\boldsymbol{t}_l)\,\boldsymbol{z}_l = \sum_{k,l=1}^p\sum_{i,j\in\mathcal{F}} \boldsymbol{z}_k^\top D_{i,j}\,\boldsymbol{z}_l\, t_{k,i}\, s_{l,j} = \sum_{i,j\in\mathcal{F}} \boldsymbol{y}_i^\top D_{i,j}\,\boldsymbol{y}_j = \boldsymbol{y}^\top D\,\boldsymbol{y},$$

where

$$\boldsymbol{y}_i = \sum_{k=1}^p \boldsymbol{z}_k\, t_{k,i}, \quad \boldsymbol{y} = (\boldsymbol{y}_i)_{i\in\mathcal{F}}.$$

Since $\boldsymbol{z}_k$ and $\boldsymbol{t}_k$ are arbitrary, $\boldsymbol{y}$ is also arbitrary. Hence, $D \succcurlyeq 0$, and the chain of implications has come full circle. □

2.6.2 Supplementary lemma on the quadratic optimization problem

Proof of Lemma 2.7. For $j\in\{1,\ldots,d\}$ define

$$f_j(\boldsymbol{t}) = \widetilde{\boldsymbol{b}}(\boldsymbol{t})\,\Sigma^{-1}(\boldsymbol{t})\,\boldsymbol{e}_j, \quad g_j(\boldsymbol{t}) = \widetilde{b}_j(\boldsymbol{t}) - b_j.$$

By continuity of $\Sigma^{-1}(\boldsymbol{t})$ and the fact that $\Sigma\mapsto\widetilde{\boldsymbol{b}}$ is Lipschitz continuous, these functions are continuous and we have

$$\begin{aligned} I(\boldsymbol{t}) &= \max\{V \mid V = \{i\in\{1,\ldots,d\}\colon\ g_i(\boldsymbol{t}) = 0,\ f_i(\boldsymbol{t}) > 0\}\},\\ K(\boldsymbol{t}) &= \max\{V \mid V = \{j\in\{1,\ldots,d\}\colon\ g_j(\boldsymbol{t}) = 0,\ f_j(\boldsymbol{t}) = 0\}\},\\ L(\boldsymbol{t}) &= \max\{V \mid V = \{l\in\{1,\ldots,d\}\colon\ g_l(\boldsymbol{t}) > 0,\ f_l(\boldsymbol{t}) = 0\}\}, \end{aligned}$$

where the maximum is in the power set of $\{1, \dots, d\}$ ordered by inclusion. It follows from the quadratic optimization lemma that these three sets are disjoint and the equality $I(\boldsymbol{t}) \cup K(\boldsymbol{t}) \cup L(\boldsymbol{t}) = \{1, \dots, d\}$ holds for each $\boldsymbol{t} \in E$.

For U, $V \subset \{1, \dots, d\}$ define

$$A_U := \bigcap_{j \in V} f_j^{-1}(0) \cap \bigcap_{j \in V^c} \left(f_j^{-1}(0)\right)^c, \quad B_V := \bigcap_{j \in V} g_j^{-1}(0) \cap \bigcap_{j \in V^c} \left(g_j^{-1}(0)\right)^c, \quad C_{U,V} = A_U \cap B_V.$$

One can check the following properties of these sets:

1. $V \neq U \iff A_V \cap A_U = \varnothing \iff B_V \cap B_U = \varnothing$ and hence $(U, V) \neq (U', V') \iff C_{U,V} \neq C_{U',V'}$.
2. $(A_V)_{V \in 2^d}$, $(B_V)_{V \in 2^d}$ and $(C_{U,V})_{U,\, V \in 2^d}$ are finite covers of E.
3. A_V, B_V and $C_{U,V}$ are locally closed sets (that is, intersections of an open and a closed set).
4. $I(\boldsymbol{t}) = U^c$ on A_U.
5. $L(\boldsymbol{t}) = V^c$ on B_V.
6. $K(\boldsymbol{t}) = U \cap V$ on $C_{U,V}$.

In other words, we have

$$I(\boldsymbol{t}) = \sum_{U \in 2^d} U^c \, \mathbb{1}_{A_U}(\boldsymbol{t}), \quad L(\boldsymbol{t}) = \sum_{V \in 2^d} V^c \, \mathbb{1}_{B_V}(\boldsymbol{t}), \quad K(\boldsymbol{t}) = \sum_{U,\, V \in 2^d} U \cap V \, \mathbb{1}_{C_{U,V}}(\boldsymbol{t}).$$

Note that some of A_V's, B_V's or $C_{U,V}$'s may be empty.

Let us split 2^d in two parts $2^d = \mathbb{V}_1(\boldsymbol{t}) \cup \mathbb{V}_2(\boldsymbol{t})$ in the following way:

1. if $V \in \mathbb{V}_1(\boldsymbol{t})$, then for all $\varepsilon > 0$ holds $A_V \cap B_\varepsilon(\boldsymbol{t}) \neq \varnothing$,
2. if $V \in \mathbb{V}_2(\boldsymbol{t})$, then there exists $\varepsilon_0 = \varepsilon_0(V, \boldsymbol{t}) > 0$ such that $A_V \cap B_{\varepsilon_0}(\boldsymbol{t}) = \varnothing$.

Define

$$\varepsilon(\boldsymbol{t}) := \min_{V \in \mathbb{V}_2(\boldsymbol{t})} \varepsilon_0(V, \boldsymbol{t}).$$

If $\boldsymbol{s} \in B_\varepsilon(\boldsymbol{t})$ with $\varepsilon < \varepsilon(\boldsymbol{t})$ there exists $V \in \mathbb{V}_1(\boldsymbol{t})$ such that $\boldsymbol{s} \in A_V$ and therefore $I(\boldsymbol{s}) = V^c$. Since $j \in I^c(\boldsymbol{t}) = K(\boldsymbol{t}) \cup L(\boldsymbol{t}) = V$ if and only if $f_j(\boldsymbol{s}) = 0$, using continuity of f_j and letting $\varepsilon \to 0$ we obtain that $f_j(\boldsymbol{t}) = 0$ and therefore $K(\boldsymbol{s}) \cup L(\boldsymbol{s}) = V \subset K(\boldsymbol{t}) \cup L(\boldsymbol{t})$. The latter is equivalent to $I(\boldsymbol{t}) \subset I(\boldsymbol{s})$.

Now we proceed to the last claim of the lemma. Similarly to what we did above, split again 2^d in two parts $2^d = \mathbb{V}_1'(\boldsymbol{t}) \cup \mathbb{V}_2'(\boldsymbol{t})$, where

1. $V \in \mathbb{V}_1'(\boldsymbol{t})$ if for all $\varepsilon > 0$ holds $B_V \cap B_\varepsilon(\boldsymbol{t}) \neq \varnothing$,
2. $V \in \mathbb{V}_2'(\boldsymbol{t})$ if there exists $\varepsilon_0' = \varepsilon_0'(V, \boldsymbol{t}) > 0$ such that $B_V \cap B_{\varepsilon_0}(\boldsymbol{t}) = \varnothing$.

Define

$$\varepsilon'(\boldsymbol{t}) = \min_{V \in \mathbb{V}_2'(\boldsymbol{t})} \varepsilon_0'(V, \boldsymbol{t}).$$

Then, if $\boldsymbol{s} \in B_\varepsilon(\boldsymbol{t})$ with $\varepsilon < \varepsilon'(\boldsymbol{t})$, there exists $V \in \mathbb{V}_1'(\boldsymbol{t})$ such that $\boldsymbol{s} \in B_V$ and therefore $L(\boldsymbol{s}) = V^c$. Since $j \in L^c(\boldsymbol{s}) = I(\boldsymbol{s}) \cup K(\boldsymbol{s}) = V$ is equivalent to $g_j(\boldsymbol{s}) = 0$, letting $\varepsilon \to 0$ and using continuity of g_j we obtain that $g_j(\boldsymbol{t}) = 0$. Hence, $I(\boldsymbol{s}) \cup K(\boldsymbol{s}) = V \subset I(\boldsymbol{t}) \cup K(\boldsymbol{t})$. The latter is equivalent to $L(\boldsymbol{t}) \subset L(\boldsymbol{s})$.

From these two properties follows that if $\boldsymbol{s} \in B_\varepsilon(\boldsymbol{t})$ with $\varepsilon < \min\{\varepsilon(\boldsymbol{t}), \varepsilon'(\boldsymbol{t})\}$, then $K(\boldsymbol{s}) \subset K(\boldsymbol{t})$. □

2.6.3 Expansions

In this section, we develop some asymptotic expansions, first without assuming A2.6.

Inverse of Sigma

Lemma 2.17. *The inverse of the variance matrix* $\Sigma(\boldsymbol{t}) = R(\boldsymbol{t}, \boldsymbol{t})$ *admits the following asymptotic formula*

$$\Sigma^{-1}(\boldsymbol{t}) - \Sigma^{-1} \sim \Sigma^{-1} \left[\sum_{i=1}^{n} \left[B_{1,i}\, t_i^{\beta_i'} + B_{2,i}\, t_i^{\beta_i} \right] + \sum_{i,j \in \mathcal{F}}^{n} \widetilde{D}_{i,j}\, t_i^{\beta_i/2}\, t_j^{\beta_j/2} \right] \Sigma^{-1}, \tag{2.73}$$

where

$$B_{k,i} = A_{k,i} + A_{k,i}^\top, \quad k = 1, 2 \quad \text{and} \quad \widetilde{D}_{i,j} = A_{6,i,j} + B_{1,i}\Sigma^{-1} B_{1,j}, \tag{2.74}$$

and the error is of order

$$o\left(\sum_{i=1}^{n} t_i^{\beta_i} \right).$$

Proof of Lemma 2.17. Plugging $\boldsymbol{t} = \boldsymbol{s}$ into (2.8), we obtain

$$\Sigma - \Sigma(\boldsymbol{t}) = \sum_{i=1}^{n} \left[B_{1,i}\, t_i^{\beta_i'} + B_{2,i}\, t_i^{\beta_i} \right] + \sum_{i,\, j \in \mathcal{F}} A_{6,i,j}\, t_i^{\beta_i/2}\, t_j^{\beta_j/2} + o\left(\sum_{i=1}^{n} t_i^{\beta_i} \right) \tag{2.75}$$

with $B_{k,i}$, $k = 1, 2$ defined in (2.74). To find its expansion up to the second order, use the Neumann power series

$$\Sigma^{-1}(\boldsymbol{t}) = \Big[\Sigma - \big[\Sigma - \Sigma(\boldsymbol{t}) \big] \Big]^{-1} = \Big[I - \Sigma^{-1} \big[\Sigma - \Sigma(\boldsymbol{t}) \big] \Big]^{-1} \Sigma^{-1} = \sum_{k \geq 0} \left(\Sigma^{-1} \big[\Sigma - \Sigma(\boldsymbol{t}) \big] \right)^k \Sigma^{-1}.$$

Note that

$$\sum_{k \geq 3} \left(\Sigma^{-1} [\Sigma - \Sigma(\boldsymbol{t})] \right)^k \Sigma^{-1} = o\left(\sum_{i=1}^{n} t_i^{\beta_i} \right)$$

and the first three terms give

$$\Sigma^{-1}(\boldsymbol{t}) \sim \Sigma^{-1} + \Sigma^{-1} \big[\Sigma - \Sigma(\boldsymbol{t}) \big] \Sigma^{-1} + \Sigma^{-1} \big[\Sigma - \Sigma(\boldsymbol{t}) \big] \Sigma^{-1} \big[\Sigma - \Sigma(\boldsymbol{t}) \big] \Sigma^{-1}.$$

Plugging (2.75) into the latter, we obtain the desired result. □

Exponential prefactor

On several occasions we shall need a formula for the following quantity, which we shall refer to as *the exponential prefactor*:

$$\ln\left(\frac{\varphi_{\Sigma(\boldsymbol{\tau})}\left(u\boldsymbol{b}-u^{-1}\boldsymbol{x}\right)}{\varphi_{\Sigma}(u\,\boldsymbol{b})}\right) = -\frac{1}{2}\,\left(u\boldsymbol{b}-u^{-1}\boldsymbol{x}\right)^{\top}\Sigma^{-1}(\boldsymbol{\tau})\left(u\boldsymbol{b}-u^{-1}\boldsymbol{x}\right)+\frac{1}{2}\,u^2\,\boldsymbol{b}^{\top}\Sigma^{-1}\boldsymbol{b}$$

$$= -\frac{1}{2}\,u^2\,\boldsymbol{b}^{\top}\left[\Sigma^{-1}(\boldsymbol{\tau})-\Sigma^{-1}\right]\boldsymbol{b}+\boldsymbol{b}^{\top}\Sigma^{-1}(\boldsymbol{\tau})\,\boldsymbol{x}-\frac{1}{2u^2}\,\boldsymbol{x}^{\top}\Sigma^{-1}(\boldsymbol{\tau})\,\boldsymbol{x} \tag{2.76}$$

$$\leq -\frac{1}{2}\,u^2\,\boldsymbol{b}^{\top}\left[\Sigma^{-1}(\boldsymbol{\tau})-\Sigma^{-1}\right]\boldsymbol{b}+\boldsymbol{b}^{\top}\Sigma^{-1}(\boldsymbol{\tau})\,\boldsymbol{x}. \tag{2.77}$$

Using (2.73) and observing that

$$\boldsymbol{b}^{\top}\Sigma^{-1}B_{1,i}\Sigma^{-1}\boldsymbol{b} = \boldsymbol{w}^{\top}B_{1,i}\boldsymbol{w} = \boldsymbol{w}^{\top}(A_{1,i}+A_{1,i}^{\top})\,\boldsymbol{w} = \boldsymbol{w}^{\top}\,\underbrace{A_{1,i}\,\boldsymbol{w}}_{=\boldsymbol{0}}+\underbrace{\boldsymbol{w}^{\top}A_{1,i}}_{=\boldsymbol{0}}\,\boldsymbol{w} = 0, \tag{2.78}$$

we obtain

$$\boldsymbol{b}^{\top}\left[\Sigma^{-1}(\boldsymbol{\tau})-\Sigma^{-1}\right]\boldsymbol{b} = \sum_{i,j=1}^{n}\Xi_{i,j}\,\tau_i^{\beta_i/2}\,\tau_j^{\beta_j/2}+o\left(\sum_{i=1}^{n}\tau_i^{\beta_i}\right), \tag{2.79}$$

where

$$\Xi_{i,j} = \boldsymbol{w}^{\top}\left[B_{2,i}\,\mathbb{1}_{i=j}+\widetilde{D}_{i,j}\,\mathbb{1}_{i,j\in\mathcal{F}}\right]\boldsymbol{w}. \tag{2.80}$$

Note that

$$\Xi_{i,j} = \boldsymbol{w}^{\top}\left[2\,A_{2,i}\,\mathbb{1}_{i=j}+D_{i,j}\,\mathbb{1}_{i,j\in\mathcal{F}}\right]\boldsymbol{w}, \qquad D_{i,j} = A_{6,i,j}+A_{1,i}\Sigma^{-1}A_{1,j}^{\top}, \tag{2.81}$$

because $A_1\,\boldsymbol{w}=\boldsymbol{0}$ and $\boldsymbol{w}^{\top}B_{2,i}\,\boldsymbol{w} = \boldsymbol{w}^{\top}(A_{2,i}+A_{2,i}^{\top})\,\boldsymbol{w} = 2\,\boldsymbol{w}^{\top}A_{2,i}\,\boldsymbol{w}$.

Assuming A2.6, we find that

$$\sum_{i,j\in\mathcal{F}}\boldsymbol{w}^{\top}D_{i,j}\,\boldsymbol{w}\,\tau_i^{\beta_i/2}\,\tau_j^{\beta_j/2} = \widehat{\boldsymbol{w}}^{\top}D\,\widehat{\boldsymbol{w}}\geq 0,$$

where $\widehat{\boldsymbol{w}} = (\boldsymbol{w}\,\tau_i^{\beta_i/2})_{i\in\mathcal{F}}\in\mathbb{R}^{\mathcal{F}}$. Therefore,

$$\sum_{i,j=1}^{n}\Xi_{i,j}\,\tau_i^{\beta_i/2}\,\tau_j^{\beta_j/2} = 2\sum_{i=1}^{n}\boldsymbol{w}^{\top}A_{2,i}\,\boldsymbol{w}\,\tau_i^{\beta_i}+\underbrace{\sum_{i,i\in\mathcal{F}}\boldsymbol{w}^{\top}D_{i,j}\,\boldsymbol{w}\,\tau_i^{\beta_i/2}\,\tau_j^{\beta_j/2}}_{\geq 0}\geq 2\sum_{i=1}^{n}\boldsymbol{w}^{\top}A_{2,i}\,\boldsymbol{w}\,\tau_i^{\beta_i}. \tag{2.82}$$

Generalized variance expansion

Lemma 2.18. *The generalized variance*

$$\sigma_{\boldsymbol{b}}^{-2}(\boldsymbol{\tau}) = \min_{\boldsymbol{x}\geq\boldsymbol{b}}\boldsymbol{x}^{\top}\,\Sigma^{-1}(\boldsymbol{\tau})\,\boldsymbol{x}$$

admits the following asymptotic formula:

$$\sigma_{\boldsymbol{b}}^{-2}(\boldsymbol{\tau})-\sigma_{\boldsymbol{b}}^{-2}(\boldsymbol{0}) = \sum_{i,j=1}^{n}\Xi_{i,j}\,\tau_i^{\beta_i/2}\,\tau_j^{\beta_j/2}+o\left(\sum_{i=1}^{n}\tau_i^{\beta_i}\right), \tag{2.83}$$

where Ξ *is defined by* (2.80) *or* (2.81).

Proof of Lemma 2.18. We remind the reader of our notation convention $A_I = A_{II} = (A_{ij})_{i,\,j\in I}$ for a given $d\times d$ matrix A and $I\subset\{1,\dots,d\}$.

Define a vector-valued function $\boldsymbol{b}(\boldsymbol{\tau})$ by

$$\sigma_{\boldsymbol{b}}^{-2}(\boldsymbol{\tau}) = \min_{\boldsymbol{x}\ge\boldsymbol{b}} \boldsymbol{x}^\top\,\Sigma^{-1}(\boldsymbol{\tau})\,\boldsymbol{x} =: \boldsymbol{b}^\top(\boldsymbol{\tau})\,\Sigma^{-1}(\boldsymbol{\tau})\,\boldsymbol{b}(\boldsymbol{\tau}).$$

We have

$$\sigma_{\boldsymbol{b}}^{-2}(\boldsymbol{\tau}) - \sigma_{\boldsymbol{b}}^{-2}(\boldsymbol{0}) = \boldsymbol{b}^\top(\boldsymbol{\tau})\,\Sigma^{-1}(\boldsymbol{\tau})\,\boldsymbol{b}(\boldsymbol{\tau}) - \boldsymbol{b}^\top(\boldsymbol{0})\,\Sigma^{-1}(\boldsymbol{0})\,\boldsymbol{b}(\boldsymbol{0}). \tag{2.84}$$

Adding and subtracting $\boldsymbol{b}(\boldsymbol{0})$ and using Lemma 2.5, we obtain

$$\begin{aligned}\boldsymbol{b}^\top(\boldsymbol{\tau})\,\Sigma^{-1}(\boldsymbol{\tau})\,\boldsymbol{b}(\boldsymbol{\tau}) &= \boldsymbol{b}^\top(\boldsymbol{\tau})\,\Sigma^{-1}(\boldsymbol{\tau})\,\boldsymbol{b}(\boldsymbol{0}) + \boldsymbol{b}(\boldsymbol{\tau})\,\Sigma(\boldsymbol{\tau})\,\Big[\boldsymbol{b}(\boldsymbol{\tau}) - \boldsymbol{b}(\boldsymbol{0})\Big]\\ &= \boldsymbol{b}^\top(\boldsymbol{\tau})\,\Sigma^{-1}(\boldsymbol{\tau})\,\boldsymbol{b}(\boldsymbol{0}) + \boldsymbol{b}_{I(\boldsymbol{\tau})}^\top(\boldsymbol{\tau})\,\Sigma_{I(\boldsymbol{\tau})}^{-1}(\boldsymbol{\tau})\,\Big[\boldsymbol{b}(\boldsymbol{\tau}) - \boldsymbol{b}(\boldsymbol{0})\Big]_{I(\boldsymbol{\tau})}.\end{aligned}$$

Take $\varepsilon(\boldsymbol{0})>0$ as in Lemma 2.7 and let $|\boldsymbol{\tau}|<\varepsilon(\boldsymbol{0})$. Then, we have that $I(\boldsymbol{\tau})\subset I(\boldsymbol{0})$, and therefore $\boldsymbol{b}_{I(\boldsymbol{\tau})}(\boldsymbol{\tau}) = \boldsymbol{b}_{I(\boldsymbol{\tau})} = \boldsymbol{b}_{I(\boldsymbol{\tau})}(\boldsymbol{0})$, hence the last term on the right is zero. Similarly, adding and subtracting $\boldsymbol{b}(\boldsymbol{\tau})$ in the second term of (2.84) and applying Lemma 2.5, we obtain

$$\boldsymbol{b}^\top(\boldsymbol{0})\,\Sigma^{-1}(\boldsymbol{0})\,\boldsymbol{b}(\boldsymbol{0}) = \boldsymbol{b}^\top(\boldsymbol{\tau})\,\Sigma^{-1}(\boldsymbol{0})\,\boldsymbol{b}(\boldsymbol{0}) + \Big[\boldsymbol{b}(\boldsymbol{0}) - \boldsymbol{b}(\boldsymbol{\tau})\Big]_{I(\boldsymbol{0})}^\top\,\Sigma_{I(\boldsymbol{0})}^{-1}(\boldsymbol{0})\,\boldsymbol{b}_{I(\boldsymbol{0})}(\boldsymbol{0}).$$

The last term is zero, since $I(\boldsymbol{0})\subset I(\boldsymbol{t})\cup K(\boldsymbol{t})$, and therefore $\boldsymbol{b}_{I(\boldsymbol{0})}(\boldsymbol{0}) = \boldsymbol{b}_{I(\boldsymbol{0})} = \boldsymbol{b}_{I(\boldsymbol{0})}(\boldsymbol{\tau})$. Therefore,

$$\sigma_{\boldsymbol{b}}^{-2}(\boldsymbol{\tau}) - \sigma_{\boldsymbol{b}}^{-2}(\boldsymbol{0}) = \boldsymbol{b}^\top(\boldsymbol{\tau})\,\Big[\Sigma^{-1}(\boldsymbol{\tau}) - \Sigma^{-1}(\boldsymbol{0})\Big]\,\boldsymbol{b}(\boldsymbol{0}). \tag{2.85}$$

Note the small difference with the formula (2.79): here we have $\boldsymbol{b}(\boldsymbol{\tau})$ on the left instead of $\boldsymbol{b}=\boldsymbol{b}(\boldsymbol{0})$. This, however, does not matter within the given error, since by A2.3 and (2.73) the difference

$$\Big[\boldsymbol{b}(\boldsymbol{\tau}) - \boldsymbol{b}(\boldsymbol{0})\Big]^\top\Big[\Sigma^{-1}(\boldsymbol{\tau}) - \Sigma^{-1}\Big]\,\boldsymbol{b}(\boldsymbol{0}) = o\left(\sum_{i,j=1}^n \tau_i^{\beta_i'}\,\tau_j^{\beta_j'}\right)$$

is within the required error. By (2.78), we arrive at (2.83). □

Conditional mean

Lemma 2.19. *The conditional mean vector*

$$\boldsymbol{d}_{\boldsymbol{\tau}}(\boldsymbol{t}) = \Big[I - R(\boldsymbol{\tau}+\boldsymbol{t},\boldsymbol{\tau})\,\Sigma^{-1}(\boldsymbol{\tau})\Big]\boldsymbol{b}$$

admits the following asymptotic formula

$$\begin{aligned}\boldsymbol{d}_{\boldsymbol{\tau}}(\boldsymbol{t}) = \sum_{i=1}^n &\left[A_{2,i}\Big[(\tau_i+t_i)^{\beta_i} - \tau_i^{\beta_i}\Big] + S_{\alpha_i,A_{5,i}}(t_i)\right]\boldsymbol{w}\\ &+ \sum_{i,j\in\mathcal{F}} \tau_j^{\beta_j/2}\,\Big[(\tau_i+t_i)^{\beta_i/2} - \tau_i^{\beta_i/2}\Big]D_{i,j}\,\boldsymbol{w} + \epsilon(\boldsymbol{\tau},\boldsymbol{t}),\end{aligned} \tag{2.86}$$

with $D_{i,j} = A_{6,i,j} + A_{1,i}\Sigma^{-1}A_{1,j}^\top$ and the error ϵ satisfying

$$\epsilon(\boldsymbol{\tau},\boldsymbol{t}) = o\left(\sum_{i=1}^n \Big[\tau_i^{\beta_i} + t_i^{\beta_i} + |t_i|^{\alpha_i}\Big]\right).$$

Proof of Lemma 2.19. Throughout this calculation, the equivalence relation $f \sim g$ is defined as follows:

$$f \sim g \iff f - g = o\left(\sum_{i=1}^{n}\left[t_i^{\beta_i} + \tau_i^{\beta_i} + |t_i|^{\alpha_i}\right]\right).$$

In order to simplify the computation, define C and B by

$$R(\boldsymbol{t},\boldsymbol{s}) =: \Sigma - C, \qquad \Sigma^{-1}(\boldsymbol{\tau}) =: \Sigma^{-1} + \Sigma^{-1} B\, \Sigma^{-1}.$$

With these shorthands, we have

$$I - R(\boldsymbol{\tau}+\boldsymbol{t},\boldsymbol{\tau})\,\Sigma^{-1}(\boldsymbol{\tau}) = I - \Big[\Sigma - C\Big]\Big[\Sigma^{-1} + \Sigma^{-1}B\,\Sigma^{-1}\Big] = (C-B)\,\Sigma^{-1} + C\,\Sigma^{-1}B\,\Sigma^{-1}$$

Note that

$$\begin{aligned} C \sim & \sum_{i=1}^{n}\Big[A_{1,i}\,(\tau_i+t_i)^{\beta_i'} + A_{2,i}\,(\tau_i+t_i)^{\beta_i} + A_{1,i}^{\top}\,\tau_i^{\beta_i'} + A_{2,i}^{\top}\,\tau_i^{\beta_i} + S_{\alpha_i,A_{5,i}}(t_i)\Big] \\ & + \sum_{i,\,j\in\mathcal{F}} A_{6,i,j}\,(\tau_i+t_i)^{\beta_i/2}\,\tau_j^{\beta_j/2}, \\ B \sim & \sum_{i=1}^{n}\Big[B_{1,i}\,\tau_i^{\beta_i'} + B_{2,i}\,\tau_i^{\beta_i}\Big] + \sum_{i,j\in\mathcal{F}}(A_{6,i,j} + B_{1,i}\Sigma^{-1}B_{1,j})\,\tau_i^{\beta_i/2}\,\tau_j^{\beta_j/2}, \end{aligned}$$

with $B_{k,i} = A_{k,i} + A_{k,i}^{\top}$, $k = 1,2$. First term:

$$\begin{aligned} (C-B)\,\Sigma^{-1} \sim & \sum_{i=1}^{n}\left[A_{1,i}\left[(\tau_i+t_i)^{\beta_i'} - \tau_i^{\beta_i'}\right] + A_{2,i}\left[(\tau_i+t_i)^{\beta_i} - \tau_i^{\beta_i}\right] + S_{\alpha_i,A_{5,i}}(t_i)\right]\Sigma^{-1} \\ & - \sum_{i,j\in\mathcal{F}} B_{1,i}\Sigma^{-1}B_{1,j}\Sigma^{-1}\,\tau_i^{\beta_i/2}\,\tau_j^{\beta_j/2} + \sum_{i,j\in\mathcal{F}} A_{6,i,j}\Sigma^{-1}\Big[(\tau_i+t_i)^{\beta_i/2} - \tau_i^{\beta_i/2}\Big]\,\tau_j^{\beta_j/2}. \end{aligned}$$

Second term:

$$C\,\Sigma^{-1}B\,\Sigma^{-1} \sim \sum_{i,j\in\mathcal{F}}\Big[A_{1,i}\Sigma^{-1}B_{1,j}(\tau_i+t_i)^{\beta_i/2}\tau_j^{\beta_j/2} + A_{1,i}^{\top}\Sigma^{-1}B_{1,j}\tau_i^{\beta_i/2}\,\tau_j^{\beta_j/2}\Big]\Sigma^{-1}.$$

Adding the two formulae together, we observe a few cancellations:

$$\begin{aligned} & I - R(\boldsymbol{\tau}+\boldsymbol{t},\boldsymbol{\tau})\,\Sigma^{-1}(\boldsymbol{\tau}) \\ & \sim \sum_{i=1}^{n}\left[A_{1,i}\Big[(\tau_i+t_i)^{\beta_i'} - \tau_i^{\beta_i'}\Big] + A_{2,i}\Big[(\tau_i+t_i)^{\beta_i} - \tau_i^{\beta_i}\Big] + S_{\alpha_i,A_{5,i}}(t_i)\right]\Sigma^{-1} \\ & \qquad + \sum_{i,j\in\mathcal{F}} \tau_j^{\beta_j/2}\Big[(\tau_i+t_i)^{\beta_i/2} - \tau_i^{\beta_i/2}\Big]\Big[A_{6,i,j} + A_{1,i}\Sigma^{-1}B_{1,j}\Big]\Sigma^{-1}. \end{aligned} \tag{2.87}$$

Multiplying by $\boldsymbol{b}$ on the right and using $A_{1,i}\,\boldsymbol{w} = \boldsymbol{0}$ yields (2.86). $\square$

Conditional covariance

Lemma 2.20. *The conditional covariance function*

$$\mathcal{R}_{\boldsymbol{\tau}}(\boldsymbol{t},\boldsymbol{s}) := R(\boldsymbol{\tau}+\boldsymbol{t},\boldsymbol{\tau}+\boldsymbol{s}) - R(\boldsymbol{\tau}+\boldsymbol{t},\boldsymbol{\tau})\,\Sigma^{-1}(\boldsymbol{\tau})\,R(\boldsymbol{\tau},\boldsymbol{\tau}+\boldsymbol{s})$$

admits the following asymptotic formula

$$\begin{aligned}\mathcal{R}_{\boldsymbol{\tau}}(\boldsymbol{t},\boldsymbol{s}) = \sum_{i=1}^{n} &\Big[S_{\alpha_i,A_{5,i}}(t_i) + S_{\alpha_i,A_{5,i}}(-s_i) - S_{\alpha_i,A_{5,i}}(t_i-s_i)\Big]\\ &+ \sum_{i,j\in\mathcal{F}} D_{i,j}\left((\tau_i+t_i)^{\beta_i/2} - \tau_i^{\beta_i/2}\right)\left((\tau_j+s_j)^{\beta_j/2} - \tau_j^{\beta_j/2}\right) + \epsilon(\boldsymbol{\tau},\boldsymbol{t},\boldsymbol{s}) \qquad (2.88)\end{aligned}$$

with $D_{i,j} = A_{6,i,j} + A_{1,i}\Sigma^{-1}A_{1,j}^\top$ *and the error* ϵ *satisfying*

$$\epsilon(\boldsymbol{\tau},\boldsymbol{t},\boldsymbol{s}) = o\left(\sum_{i=1}^{n}\left[t_i^{\beta_i} + s_i^{\beta_i} + \tau_i^{\beta_i} + |t_i|^{\alpha_i} + |s_i|^{\alpha_i} + |t_i-s_i|^{\alpha_i}\right]\right) \qquad (2.89)$$

as $(\boldsymbol{\tau},\boldsymbol{t},\boldsymbol{s}) \to \mathbf{0}$.

Proof of Lemma 2.20. Let us, for the sake of clarity, denote

$$R(\boldsymbol{t},\boldsymbol{s}) =: \Sigma - C(\boldsymbol{t},\boldsymbol{s}), \qquad \Sigma^{-1}(\boldsymbol{\tau}) =: \Sigma^{-1} + \Sigma^{-1}\,B(\boldsymbol{\tau})\,\Sigma^{-1}. \qquad (2.90)$$

Therefore, we have

$$\begin{aligned}\mathcal{R}(\boldsymbol{\tau},\boldsymbol{t},\boldsymbol{s}) &= \Sigma - C(\boldsymbol{\tau}+\boldsymbol{t},\boldsymbol{\tau}+\boldsymbol{s})\\ &\quad -\Big[\Sigma - C(\boldsymbol{\tau}+\boldsymbol{t},\boldsymbol{\tau})\Big]\Big[\Sigma^{-1} + \Sigma^{-1}\,B(\boldsymbol{\tau})\,\Sigma^{-1}\Big]\Big[\Sigma - C(\boldsymbol{\tau},\boldsymbol{\tau}+\boldsymbol{s})\Big]\\ &= -C(\boldsymbol{\tau}+\boldsymbol{t},\boldsymbol{\tau}+\boldsymbol{s}) + C(\boldsymbol{\tau}+\boldsymbol{t},\boldsymbol{\tau}) + C(\boldsymbol{\tau},\boldsymbol{\tau}+\boldsymbol{s}) - B(\boldsymbol{\tau}) + F + \epsilon,\end{aligned}$$

where

$$F := B(\boldsymbol{\tau})\,\Sigma^{-1}\,C(\boldsymbol{\tau},\boldsymbol{\tau}+\boldsymbol{s}) + C(\boldsymbol{\tau}+\boldsymbol{t},\boldsymbol{\tau})\,\Sigma^{-1}\,B(\boldsymbol{\tau}) - C(\boldsymbol{\tau}+\boldsymbol{t},\boldsymbol{\tau})\,\Sigma^{-1}\,C(\boldsymbol{\tau},\boldsymbol{\tau}+\boldsymbol{s})$$

and

$$\begin{aligned}\epsilon &:= -C(\boldsymbol{\tau}+\boldsymbol{t},\boldsymbol{\tau})\,\Sigma^{-1}\,B(\boldsymbol{\tau})\,\Sigma^{-1}\,C(\boldsymbol{\tau},\boldsymbol{\tau}+\boldsymbol{s})\\ &= O\left(\sum_{i=1}^{n}\left[\tau_i^{3\beta_i'} + t_i^{3\beta_i'} + s_i^{3\beta_k'} + |t_i|^{3\alpha_i} + |s_i|^{3\alpha_i}\right]\right)\end{aligned} \qquad (2.91)$$

can be subsumed into (2.89). Let us substitute C and B with their expressions

$$\begin{aligned} C(\boldsymbol{t},\boldsymbol{s}) &\sim \sum_{i=1}^{n}\left[A_{1,i}\,t_i^{\beta_i'} + A_{2,i}\,t_i^{\beta_i} + A_{1,i}^\top\,s_i^{\beta_i'} + A_{2,i}^\top\,s_i^{\beta_i} + S_{\alpha_i,A_{5,i}}(t_i-s_i)\right] + \sum_{i,j\in\mathcal{F}} A_{6,i,j}\,t_i^{\beta_i/2}\,t_j^{\beta_j/2},\\ B(\boldsymbol{\tau}) &\sim \sum_{i=1}^{n}\left[B_{1,i}\,\tau_i^{\beta_i'} + B_{2,i}\,\tau_i^{\beta_i}\right] + \sum_{i,j\in\mathcal{F}}\left[A_{6,i,j} + B_{1,i}\Sigma^{-1}B_{1,j}\right]\tau_i^{\beta_i/2}\,\tau_j^{\beta_j/2}. \end{aligned} \qquad (2.92)$$

Let us first compute the coefficients in front of various powers of τ, $\tau + t$ and $\tau + s$ in

$$-C(\boldsymbol{\tau} + \boldsymbol{t}, \boldsymbol{\tau} + \boldsymbol{s}) + C(\boldsymbol{\tau} + \boldsymbol{t}, \boldsymbol{\tau}) + C(\boldsymbol{\tau}, \boldsymbol{\tau} + \boldsymbol{s}) - B(\boldsymbol{\tau}). \tag{2.93}$$

Coefficients of the following terms are zero:

$$(\tau_i + t_i)^{\beta_i'},\ (\tau_i + t_i)^{\beta_i},\ (\tau_i + s_i)^{\beta_i'},\ (\tau_i + s_i)^{\beta_i},\ \tau_i^{\beta_i'},\ \tau_i^{\beta_i}.$$

Next, we proceed to the mixed terms:

$$\begin{aligned}
&(\tau_i + t_i)^{\beta_i/2} (\tau_j + s_j)^{\beta_j/2}\colon && -A_{6,i,j}, && \tau_i^{\beta_i/2} (\tau_j + s_j)^{\beta_j/2}\colon && A_{6,i,j},\\
&(\tau_i + t_i)^{\beta_i/2} \tau_j^{\beta_j/2}\colon && A_{6,i,j}, && \tau_i^{\beta_i/2} \tau_j^{\beta_j/2}\colon && -A_{6,i,j} - B_{1,i}\Sigma^{-1}B_{1,j}
\end{aligned}$$

Let us compare these with the corresponding orders in

$$F = B(\boldsymbol{\tau})\, \Sigma^{-1}\, C(\boldsymbol{\tau}, \boldsymbol{\tau} + \boldsymbol{s}) + C(\boldsymbol{\tau} + \boldsymbol{t}, \boldsymbol{\tau})\, \Sigma^{-1}\, B(\boldsymbol{\tau}) - C(\boldsymbol{\tau} + \boldsymbol{t}, \boldsymbol{\tau})\, \Sigma^{-1}\, C(\boldsymbol{\tau}, \boldsymbol{\tau} + \boldsymbol{s}).$$

We have:

$$\begin{aligned}
&(\tau_i + t_i)^{\beta_i/2} (\tau_j + s_j)^{\beta_j/2}\colon && -A_{1,i}\Sigma^{-1}A_{1,j}^\top,\\
&\tau^{\beta_i/2} (\tau_j + s_j)^{\beta_j/2}\colon && A_{1,i}\Sigma^{-1}A_{1,j}^\top,\\
&(\tau_i + t_i)^{\beta_i/2} \tau_j^{\beta_j/2}\colon && A_{1,i}\Sigma^{-1}A_{1,j}^\top,\\
&\tau_i^{\beta_i/2} \tau_j^{\beta_j/2}\colon && B_{1,i}\Sigma^{-1}A_{1,j} + A_{1,i}^\top\Sigma^{-1}A_{1,j}^\top.
\end{aligned}$$

Combining these, we find that the aggregate contribution of these terms to $\mathcal{R}$ is

$$\sum_{i,j \in \mathcal{F}} \left[A_{6,i,j} + A_{1,i}\Sigma^{-1}A_{1,j}^\top\right] \left((\tau_i + t_i)^{\beta_i/2} - \tau_i^{\beta_i/2}\right) \left((\tau_j + s_j)^{\beta_j/2} - \tau_j^{\beta_j/2}\right). \tag{2.94}$$

The significant S-terms appear only from (2.93). Their combined contribution is

$$\sum_{i=1}^{n} \Big[S_{\alpha_i, A_{5,i}}(t_i) + S_{\alpha_i, A_{5,i}}(-s_i) - S_{\alpha_i, A_{5,i}}(t_i - s_i)\Big]. \tag{2.95}$$

The accumulated error is of order

$$o\left(\sum_{i=1}^{n} \left[t_i^{\beta_i} + s_i^{\beta_i} + \tau_i^{\beta_i} + |t_i|^{\alpha_i} + |s_i|^{\alpha_i} + |t_i - s_i|^{\alpha_i}\right]\right).$$

□

Limiting process

Lemma 2.21. *Let $\Lambda > 0$, $\boldsymbol{S} > \boldsymbol{0}$ and let*

$$Q_u = \left\{\boldsymbol{\tau} \in \mathbb{R}_+^n \colon \tau_i = 0 \text{ if } i \in \mathcal{I}^c \text{ and } \tau_i \le u^{2/\alpha_i - 2/\nu_i}\Lambda_i/S_i \text{ if } i \in \mathcal{I}\right\}.$$

Then, the rescaled by $u^{-2/\boldsymbol{\nu}}$ contitional mean vector $\boldsymbol{d}_{u,\boldsymbol{\tau}}$ converges uniformly

$$\lim_{u \to \infty} \sup_{\boldsymbol{\tau} \in Q_u} \sup_{\boldsymbol{t} \in [\boldsymbol{0}, \boldsymbol{S}]} \left|u^2\, \boldsymbol{d}_{u,\boldsymbol{\tau}}\,(\boldsymbol{t}) - \boldsymbol{d}(\boldsymbol{t})\right| = 0$$

to the following limit

$$\boldsymbol{d}(\boldsymbol{t}) = \sum_{i\in\mathcal{I}\cup\mathcal{J}} S_{\alpha_i,A_{5,i}}(t_i)\,\boldsymbol{w} + \sum_{i\in\mathcal{J}\cup\mathcal{K}} A_{2,i}\,\boldsymbol{w}\,t_i^{\beta_i}. \tag{2.96}$$

The rescaled conditional covariance matrix $\mathcal{R}_{u,\boldsymbol{\tau}}$ *converges uniformly*

$$\lim_{u\to\infty} \sup_{\boldsymbol{\tau}\in Q_u} \sup_{\boldsymbol{t},\,\boldsymbol{s}\in[\boldsymbol{0},\boldsymbol{S}]} \left\| u^2\,\mathcal{R}_{u,\boldsymbol{\tau}}\,(\boldsymbol{t},\boldsymbol{s}) - \mathcal{R}(\boldsymbol{t},\boldsymbol{s})\right\| = 0$$

to the following limit:

$$\mathcal{R}(\boldsymbol{t},\boldsymbol{s}) = \sum_{i\in\mathcal{I}\cup\mathcal{J}} R_{\alpha_i,A_{5,i}}(t_i,s_i) + \sum_{i,j\in(\mathcal{J}\cup\mathcal{K})\cap\mathcal{F}} D_{i,j}\,t_i^{\beta_i/2}\,s_j^{\beta_j/2}, \tag{2.97}$$

where $R_{\alpha,V}$ *is defined in* (2.10) *and* $D_{i,j}$ *in* (2.81)*.*

Assuming A2.6 and using Lemma 2.16 we find that the covariance (2.97) corresponds to the following Gaussian random field:

$$\boldsymbol{Y}(\boldsymbol{t}) = \sum_{i\in\mathcal{I}\cup\mathcal{J}} \boldsymbol{Y}_{\alpha_i,A_{5,i}}(t_i) + \sum_{k\in\mathcal{F}} C_k(\boldsymbol{t})\,\boldsymbol{\mathcal{N}}_k, \qquad C_k(\boldsymbol{t}) = \sum_{i\in\mathcal{F}} C_{i,k}\,t_i^{\beta_i/2},$$

where

1. $\boldsymbol{Y}_{\alpha_i,A_{5,i}}$ are operator fractional Brownian motions associated to $R_{\alpha_i,A_{5,i}}$, independent of each other,
2. $\boldsymbol{\mathcal{N}}_k$ are standard Gaussian vectors, independent of each other and of $\boldsymbol{Y}_{\alpha_i,A_{5,i}}$,
3. $(C_{i,k})_{i,k\in\mathcal{F}}$, is a family of $d\times d$ matrices satisfying (2.9), whose existence is guaranteed by A2.6 and Lemma 2.16.

2.6.4 Calculations from the log-layer lemma

Proof of (2.40). By the definition of $\boldsymbol{\chi}_{u,\boldsymbol{k}}$ as the condional process

$$\boldsymbol{\chi}_{u,\boldsymbol{k}}(\boldsymbol{t}) = u\left(\boldsymbol{X}_{u,\boldsymbol{k}}(\boldsymbol{t}) - \boldsymbol{X}_{u,\boldsymbol{k}}(\boldsymbol{0}) \;\middle|\; \boldsymbol{X}_{u,\boldsymbol{k}}(\boldsymbol{0}) = u\boldsymbol{b} - \frac{\boldsymbol{x}}{u}\right)$$

we have

$$\mathbb{E}\left\{\boldsymbol{\chi}_{u,\boldsymbol{k}}(\boldsymbol{t})\right\} = -u\Big[I - R_u(\boldsymbol{\Lambda k}+\boldsymbol{t},\boldsymbol{\Lambda k})\,\Sigma_u^{-1}(\boldsymbol{\Lambda k})\Big]\left(u\,\boldsymbol{b} - \frac{\boldsymbol{x}}{u}\right) \tag{2.98}$$

$$= -u^2\,\boldsymbol{d}_{u,\boldsymbol{\Lambda k}}(\boldsymbol{t}) + \Big[I - R_u(\boldsymbol{\Lambda k}+\boldsymbol{t},\boldsymbol{\Lambda k})\,\Sigma_u^{-1}(\boldsymbol{\Lambda k})\Big]\boldsymbol{x}. \tag{2.99}$$

By 2.87, we have that for every $\varepsilon>0$ exists $\delta>0$ such that if $|\boldsymbol{\tau}|$, $|\boldsymbol{t}|<\delta$, then

$$\left\|I - R(\boldsymbol{\tau}+\boldsymbol{t},\boldsymbol{\tau})\,\Sigma^{-1}(\boldsymbol{\tau})\right\| \le \varepsilon$$

Setting $\boldsymbol{\tau} \rightsquigarrow u^{-2/\boldsymbol{\nu}}\boldsymbol{\Lambda k}$ and $\boldsymbol{t} \rightsquigarrow u^{-2/\boldsymbol{\nu}}\boldsymbol{t}$, with $\boldsymbol{k}$ now belonging to Q_u defined by (2.5.1), and noting that both belong to a shrinking, as $u\to\infty$, vicinity of zero, we obtain the following result: for every $\varepsilon>0$ there exists u_0 such that for all $u\ge u_0$ holds

$$\left|-u\Big[I - R(\boldsymbol{\tau}+\boldsymbol{t},\boldsymbol{\tau})\,\Sigma^{-1}(\boldsymbol{\tau})\Big]\,\frac{\boldsymbol{x}}{u}\right| \le \varepsilon\sum_{j=1}^{d}|x_j|$$

Next, we want to study the other term of (2.98). By (2.86),

$$\boldsymbol{d}_{\boldsymbol{\tau}}(\boldsymbol{t}) \sim \sum_{i=1}^{n} \Bigg[A_{2,i} \Big[(\tau_i + t_i)^{\beta_i} - \tau_i^{\beta_i} \Big] + S_{\alpha_i, A_{5,i}}(t_i) \Bigg] \boldsymbol{w}$$
$$+ \sum_{i,j \in \mathcal{F}} \tau_j^{\beta_j/2} \Big[(\tau_i + t_i)^{\beta_i/2} - \tau_i^{\beta_i/2} \Big] \Big[A_{6,i,j} + A_{1,i} \Sigma^{-1} A_{1,j}^{\top} \Big] \boldsymbol{w} + \epsilon(\boldsymbol{\tau}, \boldsymbol{t})$$

with ϵ satisfying the following condition:

$$\begin{gathered} \text{for every } \varepsilon > 0, \text{ there exists } \delta > 0 \text{ such that if } |\boldsymbol{\tau}|,\ |\boldsymbol{t}| < \delta, \text{ then} \\ |\epsilon(\boldsymbol{\tau}, \boldsymbol{t})| \le \varepsilon \sum_{i=1}^{n} \Big[\tau_i^{\beta_i} + t_i^{\beta_i} + |t_i|^{\alpha_i} \Big]. \end{gathered} \tag{2.100}$$

Bounding the mixed terms sum by

$$2\, \tau_j^{\beta_j/2} \Big[(\tau_i + t_i)^{\beta_i/2} - \tau_i^{\beta_i/2} \Big] \le \varepsilon\, \tau_j^{\beta_j} + \varepsilon^{-1} \Big[(\tau_i + t_i)^{\beta_i/2} - \tau_i^{\beta_i/2} \Big]^2$$

with the same ε as above, we obtain the following inequality:

$$|\boldsymbol{d}_{\boldsymbol{\tau}}(\boldsymbol{t})| \le c \sum_{i=1}^{n} \Bigg[\Big[(\tau_i + t_i)^{\beta_i} - \tau_i^{\beta_i} \Big] + \varepsilon^{-1} \Big[(\tau_i + t_i)^{\beta_i/2} - \tau_i^{\beta_i/2} \Big]^2 + |t_i|^{\alpha_i} \Bigg] + \varepsilon \sum_{i=1}^{n} \Big[\tau_i^{\beta_i} + t_i^{\beta_i} \Big].$$

Set again $\boldsymbol{\tau} \rightsquigarrow u^{-2/\boldsymbol{\nu}}\, \boldsymbol{\Lambda} \boldsymbol{k}$ and $\boldsymbol{t} \rightsquigarrow u^{-2/\boldsymbol{\nu}}\, \boldsymbol{t}$. New $\boldsymbol{k}$ and $\boldsymbol{t}$ belong to Q_u and $[\boldsymbol{0}, \boldsymbol{\Lambda}]$ correspondingly. Using the following inequality

$$(x+1)^{\beta} - x^{\beta} \le c\, (x \vee 1)^{\beta - 1} \quad \text{if} \quad x \ge 0, \tag{2.101}$$

which is valid with some constant $c > 0$, we obtain that

$$(\Lambda_i\, k_i + t_i)^{\zeta} - (\Lambda_i\, k_i)^{\zeta} \le \Big[(\Lambda_i\, k_i + \Lambda_i)^{\zeta} - (\Lambda_i\, k_i)^{\zeta} \Big] \le c_2\, \Lambda_i^{\zeta}\, (k_i \vee 1)^{\zeta - 1}$$

for $\zeta = \beta_i$ or $\zeta = \beta_i/2$. Therefore, the first condition of (2.39) holds with

$$G := c_3 \sum_{i=1}^{n} \Big[u^{2 - 2\beta_i/\nu_i} \Lambda_i^{\beta_i} \Big[(k_i \vee 1)^{\beta_i - 1} + \varepsilon^{-1}\, (k_i \vee 1)^{\beta_i - 2} + \varepsilon \Big(k_i^{\beta_i} + 1 \Big) \Big] + u^{2 - 2\alpha_i/\nu_i} \Lambda_i^{\alpha_i} \Big],$$

where ε can be made as small as required by taking u large enough.

Next, we seek for a bound of the variance of $\boldsymbol{w}_F^{\top}\, \boldsymbol{\chi}_{u,\boldsymbol{k},F}(\boldsymbol{t})$. We have:

$$\operatorname{Var} \Big\{ \boldsymbol{w}_F^{\top}\, \boldsymbol{\chi}_{u,\boldsymbol{k},F}(\boldsymbol{t}) \Big\} = \operatorname{Var} \Bigg\{ \sum_{j \in F} w_j \chi_{u,\boldsymbol{k},j}(\boldsymbol{t}) \Bigg\} = \sum_{j \in F} w_j^2\, \big[K_{u,\boldsymbol{k}}(\boldsymbol{t}, \boldsymbol{t}) \big]_{jj},$$

where $K_{u,\boldsymbol{k}}$ is the covariance of $\boldsymbol{\chi}_{u,\boldsymbol{k}}(\boldsymbol{t})$:

$$K_{u,\boldsymbol{k}}(\boldsymbol{t}, \boldsymbol{s}) = \mathbb{E} \Big\{ \Big[\boldsymbol{\chi}_{u,\boldsymbol{k}}(\boldsymbol{t}) - \boldsymbol{d}_{u,\boldsymbol{k}}(\boldsymbol{t}) \Big] \Big[\boldsymbol{\chi}_{u,\boldsymbol{k}}(\boldsymbol{s}) - \boldsymbol{d}_{u,\boldsymbol{k}}(\boldsymbol{s}) \Big]^{\top} \Big\} = u^2\, \mathcal{R}_{u, \boldsymbol{\Lambda} \boldsymbol{k}}(\boldsymbol{t}, \boldsymbol{s}).$$

By Lemma 2.20, we have that

$$\mathcal{R}_{\boldsymbol{\tau}}(\boldsymbol{t},\boldsymbol{t}) = \sum_{i=1}^{n} \Big[S_{\alpha_i, A_{5,i}}(t_i) + S_{\alpha_i, A_{5,i}}(-t_i) \Big]$$

$$+ \sum_{i,j\in\mathcal{F}} \Big[A_{6,i,j} + A_{1,i}\Sigma^{-1}A_{1,j}^{\top} \Big] \Big((\tau_i + t_i)^{\beta_i/2} - \tau_i^{\beta_i/2} \Big) \Big((\tau_j + t_j)^{\beta_j/2} - \tau_j^{\beta_j/2} \Big) + \epsilon(\boldsymbol{\tau},\boldsymbol{t})$$

with ϵ satisfying (2.100). Setting $\boldsymbol{\tau} \rightsquigarrow u^{-2/\boldsymbol{\nu}}\boldsymbol{\Lambda}\boldsymbol{k}$ and $\boldsymbol{t} \rightsquigarrow u^{-2/\boldsymbol{\nu}}\boldsymbol{t}$, and using again (2.101), we find that the second condition of (2.39) holds with

$$\sigma^2 := c_4 \sum_{i=1}^{n} \Big[u^{2-2\beta_i/\nu_i}\,\Lambda_i^{\beta_i} \Big[(k_i \vee 1)^{\beta_i - 2} + \varepsilon \left(k_i^{\beta_i} + 1 \right) \Big] + u^{2-2\alpha_i/\nu_i}\Lambda_i^{\alpha_i} \Big],$$

where $c_4 > 0$ and ε can be made arbitrarily small by taking u to be large enough. Adding the two bounds together yields

$$G + \sigma^2 = c_5 \sum_{i=1}^{n} \left[u^{2-2\beta_i/\nu_i}\Lambda_i^{\beta_i} \Big[(k_i \vee 1)^{\beta_i - 1} + \varepsilon^{-1} (k_i \vee 1)^{\beta_i - 2} + \varepsilon \left(k_i^{\beta_i} + 1 \right) \Big] + u^{2-2\alpha_i/\nu_i}\Lambda_i^{\alpha_i} \right].$$

□

2.6.5 Calculations from the double sum lemma

The covariance function of the field $(\boldsymbol{X}(\boldsymbol{t}) + \boldsymbol{X}(\boldsymbol{s}))/2$ is given by:

$$\begin{aligned} R(\boldsymbol{t}_1, \boldsymbol{s}_1, \boldsymbol{t}_2, \boldsymbol{s}_2) &= \frac{1}{4}\,\mathbb{E}\left\{ \Big[\boldsymbol{X}(\boldsymbol{t}_1) + \boldsymbol{X}(\boldsymbol{s}_1) \Big] \Big[\boldsymbol{X}(\boldsymbol{t}_2) + \boldsymbol{X}(\boldsymbol{s}_2) \Big]^{\top} \right\} \\ &= \frac{1}{4}\Big[R(\boldsymbol{t}_1,\boldsymbol{t}_2) + R(\boldsymbol{t}_1,\boldsymbol{s}_2) + R(\boldsymbol{s}_1,\boldsymbol{t}_2) + R(\boldsymbol{s}_1,\boldsymbol{s}_2) \Big]. \end{aligned} \tag{2.102}$$

Inverse of Sigma

By (2.102), we have

$$\Sigma(\boldsymbol{t},\boldsymbol{s}) = \frac{1}{4}\Big[\Sigma(\boldsymbol{t}) + \Sigma(\boldsymbol{s}) + R(\boldsymbol{t},\boldsymbol{s}) + R(\boldsymbol{s},\boldsymbol{t}) \Big].$$

Using (2.90), we can rewrite it as follows:

$$\Sigma(\boldsymbol{t},\boldsymbol{s}) = \Sigma - \frac{1}{4}\Big[C(\boldsymbol{t},\boldsymbol{t}) + C(\boldsymbol{s},\boldsymbol{s}) + C(\boldsymbol{t},\boldsymbol{s}) + C(\boldsymbol{s},\boldsymbol{t}) \Big] =: \Sigma - B(\boldsymbol{t},\boldsymbol{s}),$$

where we have introduced one more shorthand $B(\boldsymbol{t},\boldsymbol{s})$. Similarly to what we did in Lemma 2.17,

$$\Sigma^{-1}(\boldsymbol{t},\boldsymbol{s}) - \Sigma^{-1} = \Sigma^{-1}B(\boldsymbol{t},\boldsymbol{s})\,\Sigma^{-1} + \Sigma^{-1}B(\boldsymbol{t},\boldsymbol{s})\,\Sigma^{-1}B(\boldsymbol{t},\boldsymbol{s})\,\Sigma^{-1} + \epsilon, \tag{2.103}$$

where, similarly to (2.91), ϵ can be shown to satisfy

$$\epsilon = O\left(\sum_{i=1}^{n} \Big[t_i^{3\beta_i} + s_i^{3\beta_i} + |t_i - s_i|^{3\alpha_i} \Big] \right).$$

Using (2.92), we can find an expression for $B(\boldsymbol{t},\boldsymbol{s})$:

$$\begin{aligned} B(\boldsymbol{t},\boldsymbol{s}) \sim \frac{1}{4} \sum_{i,j\in\mathcal{F}} A_{6,i,j} \Big[t_i^{\beta_i/2}\, t_j^{\beta_j/2} + s_i^{\beta_i/2}\, s_j^{\beta_j/2} + t_i^{\beta_i/2}\, s_j^{\beta_j/2} + s_i^{\beta_i/2}\, t_j^{\beta_j/2}\Big] \\ + \frac{1}{4} \sum_{i=1}^{n} \Big[2\, B_{1,i}\, t_i^{\beta_i'} + 2\, B_{2,i}\, t_i^{\beta_i} + 2\, B_{1,i}\, s_i^{\beta_i'} + 2\, B_{2,i}\, s_i^{\beta_i} \\ + S_{\alpha_i,A_{5,i}}(t_i - s_i) + S_{\alpha_i,A_{5,i}}(s_i - t_i)\Big] \end{aligned} \tag{2.104}$$

with $B_{k,i} = A_{k,i} + A_{k,i}^\top$, $k = 1, 2$ and within the same magnitude of error. The quadratic term of (2.103) reads:

$$B(\boldsymbol{t},\boldsymbol{s})\, \Sigma^{-1} B(\boldsymbol{t},\boldsymbol{s}) \sim \frac{1}{4} \sum_{i,j\in\mathcal{F}} B_{1,i}\, \Sigma^{-1} B_{1,j} \Big[t_i^{\beta_i/2}\, t_j^{\beta_j/2} + s_i^{\beta_i/2}\, s_j^{\beta_j/2} + t_i^{\beta_i/2}\, s_j^{\beta_j/2} + s_i^{\beta_i}\, t_j^{\beta_j/2}\Big].$$

Finally, we arrive at

$$\begin{aligned} \Sigma^{-1}(\boldsymbol{t},\boldsymbol{s}) - \Sigma^{-1} \sim \frac{1}{4}\, \Sigma^{-1} \sum_{i=1}^{n} \Big[2\, B_{1,i}\, t_i^{\beta_i'} + 2\, B_{2,i}\, t_i^{\beta_i} + 2\, B_{1,i}\, s_i^{\beta_i'} + 2\, B_{2,i}\, s_i^{\beta_i} \\ + S_{\alpha_i,A_{5,i}}(t_i - s_i) + S_{\alpha_i,A_{5,i}}(s_i - t_i)\Big] \Sigma^{-1} \\ + \frac{1}{4} \sum_{i,j\in\mathcal{F}} \Sigma^{-1} \Big[A_{6,i,j} + B_{1,i}\, \Sigma^{-1} B_{1,j}\Big]\, \Sigma^{-1} \\ \times \Big[t_i^{\beta_i/2}\, t_j^{\beta_j/2} + s_i^{\beta_i/2}\, s_j^{\beta_j/2} + t_i^{\beta_i/2}\, s_j^{\beta_j/2} + s_i^{\beta_i/2}\, t_j^{\beta_j/2}\Big]. \end{aligned}$$

Rewriting it in terms of $\widetilde{D}_{i,j} = A_{6,i,j} + B_{1,j}\, \Sigma^{-1} B_{1,j}$ at

$$\begin{aligned} \Sigma^{-1}(\boldsymbol{t},\boldsymbol{s}) - \Sigma^{-1} = \frac{1}{4}\, \Sigma^{-1} \sum_{i=1}^{n} \Big[2\, B_{1,i}\, t_i^{\beta_i'} + 2\, B_{2,i}\, t_i^{\beta_i} + 2\, B_{1,i}\, s_i^{\beta_i'} + 2\, B_{2,i}\, s_i^{\beta_i} \\ + S_{\alpha_i,A_{5,i}}(t_i - s_i) + S_{\alpha_i,A_{5,i}}(s_i - t_i)\Big] \Sigma^{-1} \\ + \frac{1}{4}\, \Sigma^{-1} \sum_{i,j\in\mathcal{F}} \widetilde{D}_{i,j} \Big[t_i^{\beta_i/2}\, s_j^{\beta_j/2} + s_i^{\beta_i/2}\, t_j^{\beta_j/2} \\ + t_i^{\beta_i/2}\, t_j^{\beta_j/2} + s_i^{\beta_i/2}\, s_j^{\beta_j/2}\Big] \Sigma^{-1} + \epsilon. \end{aligned} \tag{2.105}$$

The error term ϵ satisfies

$$\epsilon = o\left(\sum_{i=1}^{n} \Big[t_i^{\beta_i} + s_i^{\beta_i} + |t_i - s_i|^{\alpha_i}\Big]\right). \tag{2.106}$$

Exponential prefactor

Multiplying (2.105) by $\boldsymbol{b}$ on both sides, and using the fact that by A2.3

$$\boldsymbol{w}^\top B_{1,i}\, \boldsymbol{w} = 0, \quad \boldsymbol{w}^\top B_{2,i}\, \boldsymbol{w} = 2\, \boldsymbol{w}^\top A_{2,i}\, \boldsymbol{w}, \quad \boldsymbol{w}^\top \widetilde{D}_{i,j}\, \boldsymbol{w} = \Xi_{i,j},$$

where $\Xi_{i,j}$ is defined in (2.81), and also

$$\boldsymbol{w}^\top S_{\alpha,A}(t)\,\boldsymbol{w} = \boldsymbol{w}^\top A\,\boldsymbol{w}\,\mathbb{1}_{t\ge 0}\,|t|^\alpha + \underbrace{\boldsymbol{w}^\top A^\top\,\boldsymbol{w}}_{=\boldsymbol{w}^\top A\,\boldsymbol{w}}\,\mathbb{1}_{t<0}\,|t|^\alpha = \boldsymbol{w}^\top A\,\boldsymbol{w}\,|t|^\alpha,$$

we obtain

$$\begin{aligned}\boldsymbol{b}^\top\Big[\Sigma^{-1}(\boldsymbol{\tau},\boldsymbol{\lambda}) - \Sigma^{-1}\Big]\boldsymbol{b} \sim \sum_{i=1}^n \Big[\boldsymbol{w}^\top A_{2,i}\,\boldsymbol{w}\,\Big[\tau_i^{\beta_i} + \lambda_i^{\beta_i}\Big] + \frac{\boldsymbol{w}^\top A_{5,i}\,\boldsymbol{w}}{2}\,|\lambda_i - \tau_i|^{\alpha_i}\Big] \\ + \frac{1}{4}\sum_{i,j\in\mathcal{F}} \Xi_{i,j}\Big[\tau_i^{\beta_i/2}\,\lambda_j^{\beta_j/2} + \lambda_i^{\beta_i/2}\,\tau_j^{\beta_j/2} + \tau_i^{\beta_i/2}\,\tau_j^{\beta_j/2} + \lambda_i^{\beta_i/2}\,\lambda_j^{\beta_j/2}\Big] \quad (2.107)\end{aligned}$$

with the same error as in (2.106).

Conditional mean: formula

The conditional mean vector $\boldsymbol{d}_{\boldsymbol{\tau},\boldsymbol{\lambda}}(\boldsymbol{t},\boldsymbol{s})$ of the field $(\boldsymbol{X}(\boldsymbol{t}) + \boldsymbol{X}(\boldsymbol{s}))/2$ is given by

$$\boldsymbol{d}_{\boldsymbol{\tau},\boldsymbol{\lambda}}(\boldsymbol{t},\boldsymbol{s}) = \Big[I - R(\boldsymbol{\tau}+\boldsymbol{t},\boldsymbol{\lambda}+\boldsymbol{s},\boldsymbol{\tau},\boldsymbol{\lambda})\,\Sigma^{-1}(\boldsymbol{\tau},\boldsymbol{\lambda})\Big]\boldsymbol{b},$$

where $R(\boldsymbol{t}_1,\boldsymbol{s}_1,\boldsymbol{t}_2,\boldsymbol{s}_2)$ is defined by (2.102).

As in the proof of 2.19, to simplify the computations define C and B by

$$R(\boldsymbol{\tau}+\boldsymbol{t},\boldsymbol{\lambda}+\boldsymbol{s},\boldsymbol{\tau},\boldsymbol{\lambda}) =: \Sigma - C, \qquad \Sigma^{-1}(\boldsymbol{\tau},\boldsymbol{\lambda}) =: \Sigma^{-1} + \Sigma^{-1}\,B\,\Sigma^{-1}.$$

With these shorthands, we have

$$\begin{aligned}I - R(\boldsymbol{\tau}+\boldsymbol{t},\boldsymbol{\lambda}+\boldsymbol{s},\boldsymbol{\tau},\boldsymbol{\lambda})\,\Sigma^{-1}(\boldsymbol{\tau},\boldsymbol{\lambda}) &= I - \Big[\Sigma - C\Big]\Big[\Sigma^{-1} + \Sigma^{-1}B\,\Sigma^{-1}\Big] \\ &= (C-B)\,\Sigma^{-1} + C\,\Sigma^{-1}B\,\Sigma^{-1}.\end{aligned} \quad (2.108)$$

By (2.103),

$$C = \frac{1}{4}\Big[C(\boldsymbol{\tau}+\boldsymbol{t},\boldsymbol{\tau}) + C(\boldsymbol{\tau}+\boldsymbol{t},\boldsymbol{\lambda}) + C(\boldsymbol{\lambda}+\boldsymbol{s},\boldsymbol{\tau}) + C(\boldsymbol{\lambda}+\boldsymbol{s},\boldsymbol{\lambda})\Big]$$

with

$$C(\boldsymbol{t},\boldsymbol{s}) \sim \sum_{i=1}^n \Big[A_{1,i}\,t^{\beta_i'} + A_{2,i}\,t^{\beta_i} + A_{1,i}^\top\,s_i^{\beta_i'} + A_{2,i}^\top\,s_i^{\beta_i} + S_{\alpha_i,A_{5,i}}(t_i - s_i)\Big] + \sum_{i,\,j\in\mathcal{F}} A_{6,i,j}\,t_i^{\beta_i/2}\,s_j^{\beta_j/2},$$

and

$$\begin{aligned}B \sim \frac{1}{4}\sum_{i=1}^n \Big[2\,B_{1,i}\,\tau_i^{\beta_i'} + 2\,B_{2,i}\,\tau_i^{\beta_i} + 2\,B_{1,i}\,\lambda_i^{\beta_i'} + 2\,B_{2,i}\,\lambda_i^{\beta_i} + S_{\alpha_i,A_{5,i}}(\tau_i - \lambda_i) + S_{\alpha_i,A_{5,i}}(\tau_i - \lambda_i)\Big] \\ + \frac{1}{4}\sum_{i,j\in\mathcal{F}} \widetilde{D}_{i,j}\,\Big[\tau_i^{\beta_i/2}\,\tau_j^{\beta_j/2} + \tau_i^{\beta_i/2}\,\lambda_j^{\beta_j/2} + \lambda_i^{\beta_i/2}\,\tau_j^{\beta_j/2} + \lambda_i^{\beta_i/2}\,\lambda_j^{\beta_j/2}\Big]\end{aligned}$$

Let us calculate the leading order coefficients in (2.108). First, the S-type contributions are

$$F_1 := \frac{1}{4}\sum_{i=1}^n \Big[2\,S_{\alpha_i,A_{5,i}}(t_i) + 2\,S_{\alpha_i,A_{5,i}}(s_i) + S_{\alpha_i,A_{5,i}}(\tau_i + t_i - \lambda_i)$$

$$+ S_{\alpha_i,A_{5,i}}(\lambda_i + s_i - \tau_i) - S_{\alpha_i,A_{5,i}}(\tau_i - \lambda_i) - S_{\alpha_i,A_{5,i}}(\lambda_i - \tau_i)\Big]\Sigma^{-1}. \quad (2.109)$$

Next, the leading power-type orders give

$$(\tau_i + t_i)^{\beta_i'} - \tau_i^{\beta_i'},\ (\lambda_i + s_i)^{\beta_i'} - \lambda_i^{\beta_i} \quad : \quad \frac{1}{2}\, A_{1,i},$$

$$(\tau_i + t_i)^{\beta_i} - \tau_i^{\beta_i},\ (\lambda_i + s_i)^{\beta_i} - \lambda_i^{\beta_i} \quad : \quad \frac{1}{2}\, A_{2,i}.$$

It remains to compute the mixed terms: from $C - D$:

$$\begin{matrix} (\tau_i + t_i)^{\beta_i/2}\, \tau_j^{\beta_j/2}, & (\lambda_i + s_i)^{\beta_i/2}\, \lambda_j^{\beta_j/2}, \\ (\tau_i + t_i)^{\beta_i/2}\, \lambda_j^{\beta_j/2}, & (\lambda_i + s_i)^{\beta_i/2}\, \tau_j^{\beta_j/2} \end{matrix} \quad : \quad \frac{1}{4}\, A_{6,i,j} \tag{2.110}$$

and

$$\begin{matrix} \tau_i^{\beta_i/2}\, \tau_j^{\beta_j/2}, & \tau_i^{\beta_i/2}\, \lambda_j^{\beta_j/2}, \\ \lambda_i^{\beta_i/2}\, \tau_j^{\beta_j/2}, & \lambda_i^{\beta_i/2}\, \lambda_j^{\beta_j/2} \end{matrix} \quad : \quad -\frac{1}{4}\, \widetilde{D}_{i,j}. \tag{2.111}$$

From $C\, \Sigma^{-1} B$:

$$\begin{matrix} (\tau_i + t_i)^{\beta_i/2}\, \tau_j^{\beta_j/2}, & (\tau_i + t_i)^{\beta_i/2}\, \lambda_j^{\beta_j/2}, \\ (\lambda_i + s_i)^{\beta_i/2}\, \tau_j^{\beta_j/2}, & (\lambda_i + s_i)^{\beta_i/2}\, \lambda_j^{\beta_j/2} \end{matrix} \quad : \quad \frac{1}{4}\, A_{1,i}\, \Sigma^{-1} B_{1,j} \tag{2.112}$$

and

$$\begin{matrix} \tau_i^{\beta_i/2}\, \tau_j^{\beta_j/2}, & \tau_i^{\beta_i/2}\, \lambda_j^{\beta_j/2}, \\ \lambda_i^{\beta_i/2}\, \tau_j^{\beta_j/2}, & \lambda_i^{\beta_i/2},\, \lambda_j^{\beta_j/2} \end{matrix} \quad : \quad \frac{1}{4}\, A_{1,i}^{\top}\, \Sigma^{-1} B_{1,j}. \tag{2.113}$$

Combining (2.110) and (2.112) gives

$$\begin{matrix} (\tau_i + t_i)^{\beta_i/2}\, \tau_j^{\beta_j/2}, & (\tau_i + t_i)^{\beta_i/2}\, \lambda_j^{\beta_j/2}, \\ (\lambda_i + s_i)^{\beta_i/2}\, \tau_j^{\beta_j/2}, & (\lambda_i + s_i)^{\beta_i/2}\, \lambda_j^{\beta_j/2} \end{matrix} \quad : \quad \frac{1}{4}\, A_{6,i,j} + \frac{1}{4}\, A_{1,i}\, \Sigma^{-1} B_{1,j}.$$

Similarly for (2.111) and (2.113):

$$\begin{matrix} \tau_i^{\beta_i/2}\, \tau_j^{\beta_j/2}, & \tau_i^{\beta_i/2}\, \lambda_j^{\beta_j/2}, \\ \lambda_i^{\beta_i/2}\, \tau_j^{\beta_j/2}, & \lambda_i^{\beta_i/2},\, \lambda_j^{\beta_j/2} \end{matrix} \quad : \quad -\frac{1}{4}\, A_{6,i,j} - \frac{1}{4}\, A_{1,i}\, \Sigma^{-1} B_{1,j},$$

where we have used

$$-\widetilde{D}_{i,j} + A_{1,i}^{\top}\, \Sigma^{-1} B_{1,j} = -A_{6,i,j} - B_{1,i}\, \Sigma^{-1} B_{1,j} + A_{1,i}^{\top}\, \Sigma^{-1} B_{1,j} = -A_{6,i,j} - A_{1,i}\, \Sigma^{-1} B_{1,j}$$

The aggregate power-type contribution is

$$\begin{aligned} F_2 &:= \frac{1}{2} \sum_{i=1}^{n} \Bigg[A_{1,i} \Big[\Big((\tau_i + t_i)^{\beta_i'} - \tau_i^{\beta_i'} \Big) + \Big((\lambda_i + s_i)^{\beta_i'} - \lambda_i^{\beta_i'} \Big) \Big] \\ &\quad + A_{2,i} \Big[\Big((\tau_i + t_i)^{\beta_i} - \tau_i^{\beta_i} \Big) + \Big((\lambda_i + s_i)^{\beta_i} - \lambda_i^{\beta_i} \Big) \Big] \Bigg] \Sigma^{-1} \\ &\quad + \frac{1}{4} \sum_{i,j \in \mathcal{F}} \Big[A_{6,i,j} + A_{1,i}\, \Sigma^{-1} B_{1,j} \Big] \Sigma^{-1} \Big(\tau_j^{\beta_j/2} + \lambda_j^{\beta_j/2} \Big) \\ &\quad \times \Bigg[\Big((\tau_i + t_i)^{\beta_i/2} - \tau_i^{\beta_i/2} \Big) + \Big((\lambda_i + s_i)^{\beta_i/2} - \lambda_i^{\beta_i/2} \Big) \Bigg] \end{aligned} \tag{2.114}$$

We have thus shown that

$$I - R(\boldsymbol{\tau}+\boldsymbol{t},\boldsymbol{\lambda}+\boldsymbol{s},\boldsymbol{\tau},\boldsymbol{\lambda})\,\Sigma^{-1}(\boldsymbol{\tau},\boldsymbol{\lambda}) = F_1 + F_2 + \epsilon \tag{2.115}$$

with F_2 defined in (2.114), F_1 defined in (2.109) and the error ϵ of order

$$o\left(\sum_{i=1}^{n}\Big[\tau_i^{\beta_i}+\lambda_i^{\beta_i}+t_i^{\beta_i}+s_i^{\beta_i}+|t_i|^{\alpha_i}+|s_i|^{\alpha_i}+|t_i-s_i|^{\alpha_i}+|\tau_i+t_i-\lambda_i|^{\alpha_i}+|\lambda_i+s_i-\tau_i|^{\alpha_i}\Big]\right). \tag{2.116}$$

Conditional mean: upper bound

Recall that we are interested in the upper bound for the rescaled conditional mean vector $\boldsymbol{d}_{u,\boldsymbol{\tau},\boldsymbol{\lambda}}$ uniform in $\boldsymbol{t}$, $\boldsymbol{s}\in[\boldsymbol{0},\boldsymbol{S}]$ and $\boldsymbol{\tau}$ and $\boldsymbol{\lambda}$ such that with some $\mathcal{I}'\subset\mathcal{I}$

$$\begin{gathered}\boldsymbol{0}\le\boldsymbol{\tau}\le\boldsymbol{\lambda}\le u^{2/\boldsymbol{\nu}-2/\boldsymbol{\beta}}\boldsymbol{\Lambda}/\boldsymbol{S},\quad \boldsymbol{\tau}_{\mathcal{J}\cup\mathcal{K}}=\boldsymbol{\lambda}_{\mathcal{J}\cup\mathcal{K}}=\boldsymbol{0}_{\mathcal{J}\cup\mathcal{K}},\\ \boldsymbol{\tau}_{\mathcal{I}'}=\boldsymbol{\lambda}_{\mathcal{I}'},\quad \boldsymbol{\tau}_{\mathcal{I}\setminus\mathcal{I}'}+\mathcal{S}_{\mathcal{I}\setminus\mathcal{I}'}<\boldsymbol{\lambda}_{\mathcal{I}'\setminus\mathcal{I}}.\end{gathered} \tag{2.117}$$

Let us bound $F_1\,\boldsymbol{w}$ and $F_2\,\boldsymbol{w}$ separately.

Bound for $F_1\,\boldsymbol{w}$. Multiplying F_1 by $\boldsymbol{w}$ on the right and rescaling all time parameters by $u^{-2/\boldsymbol{\nu}}$, we find that

$$\begin{aligned}u^2\,F_{1,u}\,\boldsymbol{w} := \frac{1}{4}\sum_{i=1}^{n}u^{2-2\alpha_i/\nu_i}\Big[&2\,S_{\alpha_i,A_{5,i}}(t_i)+2\,S_{\alpha_i,A_{5,i}}(s_i)+S_{\alpha_i,A_{5,i}}(\tau_i+t_i-\lambda_i)\\ &+S_{\alpha_i,A_{5,i}}(\lambda_i+s_i-\tau_i)-S_{\alpha_i,A_{5,i}}(\tau_i-\lambda_i)-S_{\alpha_i,A_{5,i}}(\lambda_i-\tau_i)\Big]\,\boldsymbol{w}.\end{aligned}$$

Then, with some $c_1>0$ holds

$$\big|S_{\alpha_i,A_{5,i}}(t_i)\,\boldsymbol{w}\big|,\ \big|S_{\alpha_i,A_{5,i}}(s_i)\,\boldsymbol{w}\big|\le c_1\,S_i^{\alpha_i}$$

and, using

$$\Big||x\pm1|^{\zeta}-x^{\zeta}\Big|\le c_2\,(x\vee1)^{\zeta-1}\quad\text{for}\quad x\ge0, \tag{2.118}$$

we find that

$$\begin{aligned}\Big|\Big[S_{\alpha_i,A_{5,i}}(\tau_i+t_i-\lambda_i)-S_{\alpha_i,A_{5,i}}(\tau_i-\lambda_i)\Big]\,\boldsymbol{w}\Big| &\le c_3\Big[(\lambda_i-\tau_i-t_i)^{\alpha_i}-(\lambda_i-\tau_i)^{\alpha_i}\Big]\\ &\le c_4\,S_i\,\left((\lambda_i-\tau_i)\vee S_i\right)^{\alpha_i-1},\\ \Big|\Big[S_{\alpha_i,A_{5,i}}(\lambda_i+s_i-\tau_i)-S_{\alpha_i,A_{5,i}}(\lambda_i-\tau_i)\Big]\,\boldsymbol{w}\Big| &\le c_5\Big[(\lambda_i-\tau_i+s_i)^{\alpha_i}-(\lambda_i-\tau_i)^{\alpha_i}\Big]\\ &\le c_6\,S_i\,\left((\lambda_i-\tau_i)\vee S_i\right)^{\alpha_i-1}.\end{aligned}$$

Bound for $F_2\,\boldsymbol{w}$. Multiplying F_2 by $\boldsymbol{w}$ on the right, rescaling all time parameters by $u^{-2/\boldsymbol{\nu}}$ and using $A_{1,i}\,\boldsymbol{w}=\boldsymbol{0}$ yields

$$u^2\,F_{2,u}\,\boldsymbol{w}=\sum_{i=1}^{n}u^{2-2\beta_i/\nu_i}\,B_i+\sum_{i,j\in\mathcal{F}}u^{2-\beta_i/\nu_i-\beta_j/\nu_j}\,B_{i,j}, \tag{2.119}$$

where we have introduced two shorthands

$$B_i \coloneqq \frac{1}{2}\, A_{2,i}\, \boldsymbol{w} \left[\left((\tau_i + t_i)^{\beta_i} - \tau_i^{\beta_i} \right) + \left((\lambda_i + s_i)^{\beta_i} - \lambda_i^{\beta_i} \right) \right]$$

$$B_{i,j} \coloneqq \frac{1}{4}\, D_{i,j}\, \boldsymbol{w} \left[\left((\tau_i + t_i)^{\beta_i/2} - \tau_i^{\beta_i/2} \right) + \left((\lambda_i + s_i)^{\beta_i/2} - \lambda_i^{\beta_i/2} \right) \right] \left(\tau_i^{\beta_j/2} + \lambda_j^{\beta_j/2} \right).$$

They can all be bounded using (2.101) as follows:

$$|B_i| \le c_1 S_i\, (\lambda_i \vee S_i)^{\beta_i - 1}, \quad |B_{i,j}| \le c_1 \Big[S_i\, (\tau_i \vee S_i)^{\beta_i - 1} + S_j\, (\lambda_j \vee S_j)^{\beta_j - 1} \Big] \left(\tau_j^{\beta_j/2} + \lambda_j^{\beta_j/2} \right).$$

To simplify the bound further, let us get rid of the mixing, applying

$$2\, x\, y \le \varepsilon x + \varepsilon^{-1} y \tag{2.120}$$

to each of the terms. Hence, the right-hand side of (2.119) is at most

$$c_1 \sum_{i=1}^{n} u^{2 - 2\beta_i/\nu_i} \Big[S_i\, (\lambda_i \vee S_i)^{\beta_i - 1} + \varepsilon\, \lambda_j^{\beta_i} + \varepsilon^{-1}\, S_j^2\, (\lambda_j \vee S_j)^{\beta_j - 2} \Big].$$

Error. The error (2.116) is no larger than

$$\varepsilon \sum_{i=1}^{n} \Big[u^{2 - 2\beta_i/\nu_i}\, (\lambda_i \vee S_i)^{\beta_i} + u^{2 - 2\alpha_i/\nu_i}\, \left((\lambda_i - \tau_i) \vee S_i \right)^{\alpha_i} \Big],$$

where ε can be made arbitrarily small by choosing u large enough.

Combined bound.

$$\begin{aligned} \left| u^2\, \boldsymbol{d}_{u,\boldsymbol{\tau},\boldsymbol{\lambda}}(\boldsymbol{t},\boldsymbol{s}) \right| \le c_1 \sum_{i=1}^{n} \Bigg[& u^{2 - 2\alpha_i/\nu_i} \Big[S_i\, \left((\lambda_i - \tau_i) \vee S_i \right)^{\alpha_i - 1} + \varepsilon \left((\lambda_i - \tau_i) \vee S_i \right)^{\alpha_i} \Big] \\ & + u^{2 - 2\beta_i/\nu_i} \Big[S_i\, (\lambda_i \vee S_i)^{\beta_i - 1} + \varepsilon\, (\lambda_i \vee S_i)^{\beta_i} + \varepsilon^{-1}\, S_j^2\, (\lambda_j \vee S_j)^{\beta_j - 2} \Big] \Bigg] \end{aligned}$$

The right-hand side of this bound can be taken as G for (2.52).

Conditional covariance

Finally, we need a bound for

$$\begin{aligned} \mathcal{R}_{\boldsymbol{\tau},\boldsymbol{\lambda}}(\boldsymbol{t}_1, \boldsymbol{s}_1, \boldsymbol{t}_2, \boldsymbol{s}_2) = {} & R(\boldsymbol{\tau} + \boldsymbol{t}_1, \boldsymbol{\lambda} + \boldsymbol{s}_1, \boldsymbol{\tau} + \boldsymbol{t}_2, \boldsymbol{\lambda} + \boldsymbol{s}_2) \\ & + R(\boldsymbol{\tau} + \boldsymbol{t}_1, \boldsymbol{\lambda} + \boldsymbol{s}_1, \boldsymbol{\tau}, \boldsymbol{\lambda})\, \Sigma^{-1}(\boldsymbol{\tau}, \boldsymbol{\lambda})\, R(\boldsymbol{\tau}, \boldsymbol{\lambda}, \boldsymbol{\tau} + \boldsymbol{t}_2, \boldsymbol{\lambda} + \boldsymbol{s}_2). \end{aligned}$$

Introduce the following shorthands:

$$R(\boldsymbol{\tau} + \boldsymbol{t}_1, \boldsymbol{\lambda} + \boldsymbol{s}_1, \boldsymbol{\tau} + \boldsymbol{t}_2, \boldsymbol{\lambda} + \boldsymbol{s}_2) \eqqcolon \Sigma - C_1,$$

$$R(\boldsymbol{\tau} + \boldsymbol{t}_1, \boldsymbol{\lambda} + \boldsymbol{s}_1, \boldsymbol{\tau}, \boldsymbol{\lambda}) \eqqcolon \Sigma - C_2,$$

$$R(\boldsymbol{\tau}, \boldsymbol{\lambda}, \boldsymbol{\tau} + \boldsymbol{t}_2, \boldsymbol{\lambda} + \boldsymbol{s}_2) \eqqcolon \Sigma - C_3$$

$$\Sigma^{-1}(\boldsymbol{\tau}, \boldsymbol{\lambda}) \eqqcolon \Sigma^{-1} + \Sigma^{-1} B\, \Sigma^{-1} + \Sigma^{-1} B\, \Sigma^{-1} B\, \Sigma^{-1},$$

Using these shorthands, we find that $\mathcal{R}_{\boldsymbol{\tau},\boldsymbol{\lambda}}$ satisfies

$$\mathcal{R}_{\boldsymbol{\tau},\boldsymbol{\lambda}}(\boldsymbol{t}_1,\boldsymbol{s}_1,\boldsymbol{t}_2,\boldsymbol{s}_2) \sim G_1 + G_2, \quad \text{where} \quad \begin{cases} G_1 := -C_1 + C_2 - B + C_3, \\ G_2 := B\,\Sigma^{-1}C_3 + C_2\,\Sigma^{-1}B \\ \qquad - C_2\,\Sigma^{-1}C_3 - B\,\Sigma^{-1}B. \end{cases}$$

We shall also need the following formula:

$$C(\boldsymbol{t},\boldsymbol{s}) \sim \sum_{i=1}^{n} \Big[A_{1,i}\,t^{\beta_i'} + A_{2,i}\,t^{\beta_i} + A_{1,i}^{\top}\,s_i^{\beta_i'} + A_{2,i}^{\top}\,s_i^{\beta_i} + S_{\alpha_i,A_{5,i}}(t_i - s_i)\Big] + \sum_{i,\,j\in\mathcal{F}} A_{6,i,j}\,t_i^{\beta_i/2}\,s_j^{\beta_j/2},$$

Terms with $(\boldsymbol{t}_1,\boldsymbol{t}_2)$. In G_1, these terms come from

$$G_{1,1} = \frac{1}{4}\Big[-C(\boldsymbol{\tau}+\boldsymbol{t}_1,\boldsymbol{\tau}+\boldsymbol{t}_2) + C(\boldsymbol{\tau}+\boldsymbol{t}_1,\boldsymbol{\tau}) - C(\boldsymbol{\tau},\boldsymbol{\tau}) + C(\boldsymbol{\tau},\boldsymbol{\tau}+\boldsymbol{t}_2)\Big].$$

The following coefficients are zero:

$$(\tau_i + t_{1,i})^{\beta_i'},\ (\tau_i + t_{1,i})^{\beta_i},\ (\tau_i + t_{2,i})^{\beta_i'},\ (\tau_i + t_{2,i})^{\beta_i},\ \tau_i^{\beta_i'},\ \tau_i^{\beta_i}.$$

Mixed terms from $G_{1,1}$ and G_2:

$$\frac{1}{4}\sum_{i,j\in\mathcal{F}} D_{i,j}\Big[(\tau_i + t_{1,i})^{\beta_i/2} - \tau_i^{\beta_i/2}\Big]\Big[(\tau_j + t_{1,j})^{\beta_j/2} - \tau_j^{\beta_j/2}\Big].$$

S-terms:

$$\frac{1}{4}\sum_{i=1}^{n}\Big[-S_{\alpha_i,A_{5,i}}(t_{1,i} - t_{2,i}) + S_{\alpha_i,A_{5,i}}(t_{1,i}) + S_{\alpha_i,A_{5,i}}(-t_{2,i})\Big].$$

Rescaling everything by $u^{-2/\boldsymbol{\nu}}$, we obtain

$$u^2\,\left\|{}^{\text{terms with}}_{\boldsymbol{t}_1 \text{ and } \boldsymbol{t}_2}\right\| \le c_1 \sum_{i=1}^{n} u^{2-2\alpha_i/\nu_i}\,S_i^{\alpha_i} + c_1 \sum_{i,j\in\mathcal{F}} u^{2-\beta_i/\nu_i-\beta_j/\nu_j}\,S_i\,(\tau_i\vee S_i)^{\beta_i/2-1}\,S_j\,(\tau_j\vee S_j)^{\beta_j/2-1}.$$

We can also simplify the bound by getting rid of the mixing:

$$u^2\,\left\|{}^{\text{terms with}}_{\boldsymbol{t}_1 \text{ and } \boldsymbol{t}_2}\right\| \le c_2 \sum_{i=1}^{n}\left[u^{2-2\alpha_i/\nu_i}\,S_i^{\alpha_i} + u^{2-2\beta_i/\nu_i}S_i^2\,(\tau_i\vee S_i)^{\beta_i-2}\right].$$

Terms with $(\boldsymbol{t}_1,\boldsymbol{s}_2)$. In G_1, these terms come from

$$G_{1,2} = \frac{1}{4}\Big[-C(\boldsymbol{\tau}+\boldsymbol{t}_1,\boldsymbol{\lambda}+\boldsymbol{s}_2) + C(\boldsymbol{\tau}+\boldsymbol{t}_1,\boldsymbol{\lambda}) - C(\boldsymbol{\lambda},\boldsymbol{\lambda}) + C(\boldsymbol{\tau},\boldsymbol{\lambda}+\boldsymbol{s}_2)\Big].$$

The following coefficients are zero:

$$(\tau_i + t_{1,i})^{\beta_i'},\ (\tau_i + t_{1,i})^{\beta_i},\ (\lambda_i + s_{2,i})^{\beta_i'},\ (\lambda_i + s_{2,i})^{\beta_i},\ \lambda_i^{\beta_i'},\ \lambda_i^{\beta_i}.$$

Mixed terms from $G_{1,1}$ and G_2:

$$\frac{1}{4}\sum_{i,j\in\mathcal{F}} D_{i,j}\Big[(\tau_i + t_{1,i})^{\beta_i/2} - \tau_i^{\beta_i/2}\Big]\Big[(\lambda_j + s_{1,j})^{\beta_j/2} - \lambda_j^{\beta_j/2}\Big].$$

S-terms:

$$\frac{1}{4}\sum_{i=1}^{n}\Big[-S_{\alpha_i,A_{5,i}}(\tau_i+t_{1,i}-\lambda_i-s_{2,i})+S_{\alpha_i,A_{5,i}}(\tau_i+t_{1,i})-S_{\alpha_i,A_{5,i}}(\tau_i-\lambda_i)+S_{\alpha_i,A_{5,i}}(-\lambda_i-s_{2,i})\Big].$$

Rescaling everything by $u^{-2/\boldsymbol{\nu}}$, and getting rid of the mixed terms as above, we obtain

$$u^2\,\left\|{\scriptstyle\text{terms with}\atop \boldsymbol{t}_1\text{ and }\boldsymbol{t}_2}\right\|\le c_1\sum_{i=1}^{n}\Big[u^{2-2\alpha_i/\nu_i}\,S_i\left((\lambda_i-\tau_i)\vee S_i\right)^{\alpha_i-1}+u^{2-2\beta_i/\nu_i}\,S_i^2\left(\lambda_i\vee S_i\right)^{\beta_i-2}.\Big]$$

The remaining terms (with $(\boldsymbol{s}_1,\boldsymbol{t}_2)$ and $(\boldsymbol{s}_1,\boldsymbol{s}_2)$) may be estimated similarly.

Error. The accumulated error is at most

$$\varepsilon\sum_{i=1}^{n}\left[u^{2-2\beta_i/\nu_i}(\lambda_i\vee S_i)^{\beta_i}+u^{2-2\alpha_i/\nu_i}\,\left((\lambda_i-\tau_i)\vee S_i\right)^{\alpha_i}\right]$$

with ε, which can be made arbitrarily small by taking u large enough.

Combined bound. Combining the bounds together, we find that

$$u^2\,\|\mathcal{R}_{u,\boldsymbol{\tau},\boldsymbol{\lambda}}(\boldsymbol{t}_1,\boldsymbol{s}_1,\boldsymbol{t}_1,\boldsymbol{s}_1)\|\le c_1\sum_{i=1}^{n}\left[u^{2-2\alpha_i/\nu_i}\Big[S_i^{\alpha_i}+S_i\,\left((\lambda_i-\tau_i)\vee S_i\right)^{\alpha_i-1}+\varepsilon\,(\lambda_i-\tau_i)^{\alpha_i}\Big]\right.$$
$$\left.+\,u^{2-2\beta_i/\nu_i}\,\Big[S_i(\lambda_i\vee S_i)^{\beta_i-1}+\varepsilon(\lambda_i\vee S_i)^{\beta_i}\Big]\right].$$

We have therefore obtained that (2.52) holds with

$$G+\sigma^2=c_1\sum_{i=1}^{n}\left[u^{2-2\alpha_i/\nu_i}\,\Big[S_i^{\alpha_i}+S_i\,\left((\lambda_i-\tau_i)\vee S_i\right)^{\alpha_i-1}+\varepsilon(\lambda_i-\tau_i)^{\alpha_i}\Big]\right.$$
$$\left.+\,u^{2-2\beta_i/\nu_i}\,\Big[S_i\,(\lambda_i\vee S_i)^{\beta_i-1}+\varepsilon\,(\lambda_i\vee S_i)^{\beta_i}+\varepsilon^{-1}\,S_j^2\,(\lambda_j\vee S_j)^{\beta_j-2}\Big]\right].$$

2.6.6 Integral estimate

Proof of Lemma 2.10. Define a collection of sets $\Omega_F=\left\{\boldsymbol{x}\in\mathbb{R}^d\colon \boldsymbol{x}_F>\boldsymbol{0},\ \boldsymbol{x}_{F^c}<\boldsymbol{0}\right\}$ indexed by $F\subset\{1,\dots,d\}$ and split the integral:

$$\int_{\mathbb{R}^d}e^{\boldsymbol{w}^\top\boldsymbol{x}}\,\mathbb{P}\left\{\exists\boldsymbol{t}\in[\boldsymbol{0},\boldsymbol{\Lambda}]\colon\boldsymbol{\chi}_{\boldsymbol{x}}(\boldsymbol{t})>\boldsymbol{x}\right\}d\boldsymbol{x}=\sum_{F\in 2^d}\int_{\Omega_F}e^{\boldsymbol{w}^\top\boldsymbol{x}}\,\mathbb{P}\left\{\exists\boldsymbol{t}\in[\boldsymbol{0},\boldsymbol{\Lambda}]\colon\boldsymbol{\chi}_{\boldsymbol{x}}(\boldsymbol{t})>\boldsymbol{x}\right\}d\boldsymbol{x}\,.$$

For $\boldsymbol{x}\in\Omega_F$ the probability under the integral may be bounded as follows:

$$\begin{aligned}
&\mathbb{P}\left\{\exists\boldsymbol{t}\in[\boldsymbol{0},\boldsymbol{\Lambda}]\colon\boldsymbol{\chi}_{\boldsymbol{x}}(\boldsymbol{t})>\boldsymbol{x}\right\}\\
&\quad\le\mathbb{P}\Big\{\exists\boldsymbol{t}\in[\boldsymbol{0},\boldsymbol{\Lambda}]\colon\boldsymbol{w}_F^\top\big(\boldsymbol{\chi}_{\boldsymbol{x},F}(\boldsymbol{t})-\mathbb{E}\left\{\boldsymbol{\chi}_{\boldsymbol{x},F}(\boldsymbol{t})\right\}\big)>\boldsymbol{w}_F^\top\boldsymbol{x}_F-\boldsymbol{w}_F^\top\mathbb{E}\left\{\boldsymbol{\chi}_{\boldsymbol{x},F}(\boldsymbol{t})\right\}\Big\}\\
&\quad\le\mathbb{P}\left\{\exists\boldsymbol{t}\in[\boldsymbol{0},\boldsymbol{\Lambda}]\colon\boldsymbol{w}_F^\top\big(\boldsymbol{\chi}_{\boldsymbol{x}}-\mathbb{E}\left\{\boldsymbol{\chi}_{\boldsymbol{x}}(\boldsymbol{t})\right\}\big)>\boldsymbol{w}_F^\top\boldsymbol{x}_F-G-\varepsilon\sum_{j=1}^{d}|x_j|\right\}
\end{aligned}$$

$$= \mathbb{P}\left\{\exists \boldsymbol{t} \in [\boldsymbol{0}, \boldsymbol{\Lambda}] \colon \eta_{\boldsymbol{x},F}(\boldsymbol{t}) > r_{F,\varepsilon}(\boldsymbol{x}) - G\right\},$$

where

$$r_{F,\varepsilon}(\boldsymbol{x}) = \boldsymbol{w}_F^\top \boldsymbol{x}_F - \varepsilon \sum_{j=1}^d |x_j| \qquad \text{and} \qquad \eta_{\boldsymbol{x},F}(\boldsymbol{t}) = \boldsymbol{w}_F^\top \big(\boldsymbol{\chi}_{\boldsymbol{x},F}(\boldsymbol{t}) - \mathbb{E}\left\{\boldsymbol{x}_{\boldsymbol{x},F}(\boldsymbol{t})\right\}\big).$$

Let us split the domain Ω_F into two parts

$$\Omega_{F,+} = \{\boldsymbol{x} \in \Omega_F \colon r_{F,\varepsilon}(\boldsymbol{x}) > G\} \qquad \text{and} \qquad \Omega_{F,-} = \Omega_F \setminus \Omega_{F,+}.$$

Let us first deal with the integral over $\Omega_{F,-}$. It follows from $\boldsymbol{w}_F^\top \boldsymbol{x}_F - \varepsilon \sum_{j=1}^d |x_j| < G$ that

$$\sum_{j \in F} (w_i - \varepsilon)|x_j| - \varepsilon \sum_{j \in F^c} |x_j| < G$$

or, with $w_* = \min_{j \in F} w_j > 0$ and $\varepsilon < w_*$,

$$\varepsilon \sum_{j \in F} |x_j| \le \frac{\varepsilon G}{w_* - \varepsilon} + \frac{\varepsilon^2}{w_* - \varepsilon} \sum_{j \in F^c} |x_j|$$

Therefore, with $r = r_{F,\varepsilon}(\boldsymbol{x})$, we have

$$\begin{aligned} \boldsymbol{w}^\top \boldsymbol{x} = r + \boldsymbol{w}_{F^c}^\top \boldsymbol{x}_{F^c} + \varepsilon \sum_{j=1}^d |x_j| = r + \varepsilon \sum_{j \in F} |x_j| - \sum_{j \in F^c} (w_j - \varepsilon)|x_j| \\ \le r + \frac{\varepsilon G}{w_* - \varepsilon} - \left(w_* - \frac{\varepsilon^2}{w_* - \varepsilon} - \varepsilon\right) \sum_{j \in F^c} |x_j| \le r + \frac{\varepsilon G}{w_* - \varepsilon}, \end{aligned}$$

provided that ε is small enough. Bounding the probability under the integral by 1 and changing the variables, we obtain

$$\begin{aligned} \int_{\Omega_{F,-}} e^{\boldsymbol{w}^\top \boldsymbol{x}} \, \mathbb{P}\left\{\exists \boldsymbol{t} \in [\boldsymbol{0}, \boldsymbol{\Lambda}] \colon \boldsymbol{\chi}_{\boldsymbol{x}}(\boldsymbol{t}) > \boldsymbol{x}\right\} d\boldsymbol{x} \le \int_{\Omega_{F,-}} e^{\boldsymbol{w}^\top \boldsymbol{x}} \, d\boldsymbol{x} = \int_{-\infty}^{G} dr \int dS \, e^{\boldsymbol{w}^\top \boldsymbol{x}} \, |r|^{d-1} \\ \le \int_{-\infty}^{G} \int dS \, e^{r + \varepsilon G/(w_* - \varepsilon)} |r|^{d-1} \, dr \, dS \le c_1 e^{\varepsilon G/(w_* - \varepsilon)} \int_{-\infty}^{G} e^{(1+\varepsilon) r} \, dr = c_1 \, e^{c_2 G}. \end{aligned}$$

Next, we concentrate on the intergral over $\Omega_{F,+}$. By Piterbarg inequality, we have the following uniform in $\boldsymbol{x} \in \Omega_{F,+}$ upper bound:

$$\mathbb{P}\left\{\exists \boldsymbol{t} \in [\boldsymbol{0}, \boldsymbol{\Lambda}] \colon \eta_{\boldsymbol{x},F}(\boldsymbol{t}) > \boldsymbol{x}\right\} \le c_3 \left(\frac{r(\boldsymbol{x}) - G}{\sigma}\right)^{2/\gamma} \exp\left(-\frac{1}{2}\left(\frac{r(\boldsymbol{x}) - G}{\sigma}\right)^2\right).$$

Plugging this bound into the integral and changing the variables, we obtain

$$\begin{aligned} &\int_{\Omega_{F,+}} e^{\boldsymbol{w}^\top \boldsymbol{x}} \, \mathbb{P}\left\{\exists \boldsymbol{t} \in [\boldsymbol{0}, \boldsymbol{\Lambda}] \colon \boldsymbol{\chi}_{\boldsymbol{x}}(\boldsymbol{t}) > \boldsymbol{x}\right\} d\boldsymbol{x} \\ &\qquad \le c_3 \int_{\Omega_{F,+}} e^{\boldsymbol{w}^\top \boldsymbol{x}} \left(\frac{r(\boldsymbol{x}) - G}{\sigma}\right)^{2/\gamma} \exp\left(-\frac{1}{2}\left(\frac{r(\boldsymbol{x}) - G}{\sigma}\right)^2\right) d\boldsymbol{x} \end{aligned}$$

$$= c_3 \int_G^\infty dr \int dS\, e^{\boldsymbol{w}^\top \boldsymbol{x}} \left(\frac{r-G}{\sigma}\right)^{2/\gamma+d-1} \exp\left(-\frac{1}{2}\left(\frac{r-G}{\sigma}\right)^2\right).$$

Note that with $w^* = \max_{i=1,\dots,d} w_i$ we have

$$r = \sum_{i\in F}(w_i-\varepsilon)|x_i| - \varepsilon \sum_{i\in F^c}|x_i| \geq (w^*-\varepsilon)\sum_{i\in F}|x_i| - \varepsilon\sum_{i\in F^c}|x_i|$$

and it follows that for all $\varepsilon < w^*$ the following bound holds:

$$\varepsilon\sum_{i\in F}|x_i| \leq \frac{\varepsilon r}{w^*-\varepsilon} + \frac{\varepsilon^2}{w^*-\varepsilon}\sum_{i\in F^c}|x_i|.$$

This bound yields

$$(\boldsymbol{w},\boldsymbol{x}) = r + \varepsilon\sum_{i\in F}^{d}|x_i| - \sum_{i\in F^c}(w_i-\varepsilon)|x_i| \leq \left(1+\frac{\varepsilon}{w^*-\varepsilon}\right) r - \left(w_* - \varepsilon - \frac{\varepsilon^2}{w^*-\varepsilon}\right)\sum_{i\in F^c}|x_i|,$$

from which for small enough ε follows that $(\boldsymbol{w},\boldsymbol{x}) \leq (1+\varepsilon')r$, with $\varepsilon' = \varepsilon/(w^*-\varepsilon)$. Hence,

$$c_3 \int_G^\infty dr \int dS\, e^{\boldsymbol{w}^\top \boldsymbol{x}} \left(\frac{r-G}{\sigma}\right)^{2/\gamma+d-1} \exp\left(-\frac{1}{2}\left(\frac{r-G}{\sigma}\right)^2\right)$$

$$\leq c_4 \int_{-\infty}^\infty e^{(1+\varepsilon')r} \exp\left(-\frac{1}{2}\left(\frac{r-G}{\sigma}\right)^2\right) dr \leq c_5\, e^{c_6(G+\sigma^2)},$$

where in the last step we used the Gaussian mgf formula $\mathbb{E}\left\{e^{t\mathcal{N}(\mu,\sigma^2)}\right\} = e^{t\mu+t^2\sigma^2/2}$ with $t = 1+\varepsilon'$. $\square$

2.6.7 Double crossing: vicinity of the diagonal

Proof of Lemma 2.1. We begin the proof with the following upper bound:

$$\mathbb{P}\left\{\exists\, \boldsymbol{t} \in D_\varepsilon \colon X(t_1) > au,\ X(t_2) < -bu\right\} \leq \mathbb{P}\left\{\exists\, \boldsymbol{t} \in D_\varepsilon^+ \colon X(t_1) - X(t_2) > (a+b)\,u\right\} + \mathbb{P}\left\{\exists\, \boldsymbol{t} \in D_\varepsilon^- \colon X(t_1) - X(t_2) > (a+b)\,u\right\},$$

where

$$D_\varepsilon^+ = \left\{\boldsymbol{t} = (t,s) \in [0,T]^2 \colon t < s \leq t+\varepsilon\right\}, \qquad D_\varepsilon^- = \left\{\boldsymbol{t} = (t,s) \in [0,T]^2 \colon s < t \leq s+\varepsilon\right\}.$$

Define a Gaussian field

$$\mathcal{X}(s,l) \coloneqq X(s+l) - X(s), \qquad (s,l) \in \mathbb{T} \coloneqq [0,T]\times[0,\varepsilon]$$

and use it to coarsen the bound above:

$$\mathbb{P}\left\{\exists\, \boldsymbol{t} \in D_\varepsilon^- \colon X(t_1) - X(t_2) > (a+b)\,u\right\} \leq \mathbb{P}\left\{\exists\, (s,l) \in \mathbb{T} \colon \mathcal{X}(s,l) > (a+b)\,u\right\}.$$

The variance of this Gaussian random field is

$$\sigma^2(s,l) = \mathrm{Var}\{\mathcal{X}(s,l)\} = \mathbb{E}\left\{[X(s+l)-X(s)]^2\right\} \leq f(\varepsilon).$$

By Borell-TIS inequality (2.22), there exists $\mu > 0$ such that for all $u > \mu$

$$\mathbb{P}\left\{\exists\, (s,l) \in \mathbb{T}\colon \mathcal{X}(s,l) > (a+b)\, u\right\} \le \exp\left(-\frac{(u-\mu)^2}{2\, f(\varepsilon)}\right).$$

Since $f(\varepsilon) \to 0$ by the hypotheses of the theorem, for any $\delta > 0$ there exists some $\varepsilon > 0$ such that $f(\varepsilon) < 1/4\,\delta$. Therefore,

$$\mathbb{P}\left\{\exists\, \boldsymbol{t} \in D_\varepsilon\colon \mathcal{X}(s,l) > (a+b)\, u\right\} \le o\left(e^{-\delta u^2}\right).$$

□

2.6.8 Double crossing for stationary processes: expansions

Proof of Lemma 2.2. Consider $\boldsymbol{X}_1(\boldsymbol{t}) = (X(t_1), -X(T-t_2))^\top$. We want to find the expansion of

$$R(\boldsymbol{t},\boldsymbol{s}) = \mathbb{E}\left\{\boldsymbol{X}_1(\boldsymbol{t})\, \boldsymbol{X}_1(\boldsymbol{s})^\top\right\} = \begin{pmatrix} \rho(|t_1-s_1|) & -\rho(|T-s_2-t_1|) \\ -\rho(|T-t_2-s_1|) & \rho(|t_2-s_2|) \end{pmatrix}$$

near $\boldsymbol{t} = \boldsymbol{0}$. We have:

$$\Sigma = \begin{pmatrix} 1 & -\rho(T) \\ -\rho(T) & 1 \end{pmatrix},$$

and

$$\begin{aligned}\Sigma - R(\boldsymbol{t},\boldsymbol{s}) &= \begin{pmatrix} 1-\rho(|t_1-s_1|) & \rho(|T-s_2-t_1|)-\rho(T) \\ \rho(|T-t_2-s_1|)-\rho(T) & 1-\rho(|t_2-s_2|) \end{pmatrix} \\ &= A_{2,1}\, t_1 + A_{2,2}\, t_2 + A_{2,1}^\top\, s_2 + A_{2,2}^\top\, s_2 + A_{5,1}\, |t_1-s_1|^\alpha + A_{5,2}\, |t_2-s_2|^\alpha \\ &\quad + o\left(t_1+t_2+s_1+s_2+|t_1-s_1|^\alpha + |t_2-s_2|^\alpha\right),\end{aligned}$$

where the matrix coefficients are given by

$$A_{2,1} = A_{2,2}^\top = -\rho'(T)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A_{5,1} = \vartheta \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_{5,2} = \vartheta \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Clearly, $\alpha_1 = \alpha_2 = \alpha$. Next, we need the optimal vector

$$\boldsymbol{w} = \Sigma^{-1}(\boldsymbol{0})\,(a,b)^\top = \frac{1}{1-\rho^2(T)}\begin{pmatrix} 1 & \rho(T) \\ \rho(T) & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{1-\rho^2(T)}\begin{pmatrix} a+b\rho(T) \\ b+a\rho(T) \end{pmatrix}$$

to check whether we have correctly identified $\beta_1 = \beta_2 = 1$. That is, we need to check A2.4

$$\xi_1 = \boldsymbol{w}^\top A_{2,1}\, \boldsymbol{w} = \xi_2 = \boldsymbol{w}^\top A_{2,2}\, \boldsymbol{w} = \frac{-\rho'(T)(a+b\rho(T))(b+a\rho(T))}{(1-\rho^2(T))^2} > 0.$$

We also need to check A2.5:

$$\varkappa_1 = \boldsymbol{w}^\top A_{5,1}\, \boldsymbol{w} = \varkappa_2 = \boldsymbol{w}^\top A_{5,2}\, \boldsymbol{w} = \frac{C(b+a\rho(T))^2}{(1-\rho^2(T))^2} > 0.$$

Assumption A3 may be shown as follows:

$$\begin{aligned}\mathbb{E}\left\{|\boldsymbol{X}(\boldsymbol{t}) - \boldsymbol{X}(\boldsymbol{s})|^2\right\} &= \mathbb{E}\left\{(X(t_1)-X(s_1))^2 + (X(t_2)-X(s_2))^2\right\} \\ &= 2(1-\rho(|t_1-s_1|)) + 2(1-\rho(|t_2-s_2|)) \\ &\le c_1\left(|t_1-t_2|^\alpha + |t_2-s_2|^\alpha\right)\end{aligned}$$

with some constant $c_1 > 0$. The last inequality follows from the asymptotics of $\rho(t)$ near $t = 0$. Hence, Assumption A3 is satisfied with $\boldsymbol{\gamma} = (\alpha,\alpha)^\top$. □

2.6.9 Double crossing for fBm: minimization of generalized variance

Proof of Lemma 2.3. We begin the proof by making use of the positive homogenity of the generalized variance $\sigma_{a,b}^{-2}$. Namely, for $c > 0$ we have

$$\sigma_{a,b}^{-2}(c\boldsymbol{t}) = c^{-2H}\sigma_{a,b}^{-2}(\boldsymbol{t}).$$

Therefore, if we assume that $\boldsymbol{t} = (t_1, t_2)^\top$ lies in the lower triangle $t_1 > t_2$, we obtain from the equation above

$$\sigma_{a,b}^{-2}(t_1, t_2) = \left(\frac{t_1}{T}\right)^{-2H} \sigma_{a,b}^{-2}\left(T, \frac{Tt_2}{t_1}\right) \geq \sigma_{a,b}^{-2}\left(T, \frac{Tt_2}{t_1}\right),$$

since $(t_1/T)^{-2H} \geq 1$. Since $t_1 > t_2$, we have that $t_2' = Tt_2/t_1 \in (0, T)$ and it follows that

$$\min_{t_1 > t_2} \sigma_{a,b}^{-2}(t_1, t_2) \geq \min_{t_2' \in [0,T]} \sigma_{a,b}^{-2}(T, t_2'),$$

hence, to minimize $\boldsymbol{t} \mapsto \sigma_{a,b}^{-2}(\boldsymbol{t})$ in the lower triangle we only need to minimize $t_2 \mapsto \sigma_{a,b}^{-2}(T, t_2)$ in $t_2 \in (0, T)$. Similarly, to minimize $\boldsymbol{t} \mapsto \sigma_{a,b}^{-2}(\boldsymbol{t})$ in the upper triangle, we only need to minimize $t_1 \mapsto \sigma_{a,b}^{-2}(t_1, T)$ in $t_1 \in (0, T)$.

If $a < b$, the following trivial inequality

$$\sigma_{a,b}^{-2}(t, T) = \frac{a^2t^{2H} + 2abr(t, T) + b^2T^{2H}}{(tT)^{2H} - r^2(t, T)} > \frac{a^2T^{2H} + 2abr(t, T) + b^2t^{2H}}{(tT)^{2H} - r^2(t, T)} = \sigma_{a,b}^{-2}(T, t)$$

shows, that the minimum over lower triangle is strictly smaller than the minimum over upper triangle. Therefore, the global minimum lies in $t_1 > t_2$. Similarly, if $a > b$, the global minimum lies in $t_1 < t_2$. If $a = b$, we have two global minima.

Let us proceed to showing that the function

$$t \mapsto \sigma_{a,b}^{-2}(T, s) = \frac{a^2s^{2H} + 2abr(s, T) + b^2T^{2H}}{(sT)^{2H} - r^2(s, T)}$$

possesses a unique minimum in $t \in (0, T)$. Without loss of generality, we may rewrite this function as

$$\sigma_{a,b}^{-2}(T, s) = \frac{a^2}{T^{2H}} D_{b/a}\left(\frac{s}{T}\right), \tag{2.121}$$

where

$$D_\alpha(s) := \frac{\alpha^2 + 2\alpha f(s) + s^{2H}}{s^{2H} - f^2(s)} = \frac{(\alpha + f(s))^2}{s^{2H} - f^2(s)} + 1 \qquad s \in (0, 1),$$

where we introduced the function

$$f(s) = \frac{1}{2}\Big(s^{2H} + 1 - (1 - s)^{2H}\Big).$$

A straightforward approach would be to show that $D_\alpha'(0+) < 0$, $D_\alpha'(1-) > 0$ and that $D_\alpha'' > 0$. The first two claims are easily seen to be true, but, unfortunately, the third is false. The idea we shall employ to get around this issue is to multiply the function D_α' by an appropriately chosen and strictly positive function $U > 0$, so that the roots of $D_\alpha'U$ remained the same as the roots of D_α', but $D_\alpha'U$ became strictly increasing. We now proceed to finding such multiplier.

First, rewrite D'_α collecting α-free and α-dependent terms:

$$D'_\alpha(s) = \frac{2H(\alpha + f(s))}{(s^{2H} - f^2(s))^2}\Big(\alpha G_\alpha(s) + G_0(s)\Big), \tag{2.122}$$

where

$$G_\alpha(s) = \frac{f(s)f'(s)}{H} - s^{2H-1}, \qquad G_0(s) = \frac{f'(s)s^{2H}}{H} - f(s)s^{2H-1}.$$

We can drop the positive factor

$$\frac{2H(\alpha + f(s))}{(s^{2H} - f^2(s))^2} > 0,$$

since the roots of D'_α are that of $\alpha\, G_\alpha(s) + G_0(s)$. Unfortunately, this remainder is still non-monotone.

Proceding with the computations, we expand the derivatives and find that

$$\begin{aligned}
G_\alpha(s) &= f(s)\Big(s^{2H-1} + (1-s)^{2H-1}\Big) - s^{2H-1} \\
&= f(s)(1-s)^{2H-1} - (1-f(s))s^{2H-1} \\
&= f(s)(1-s)^{2H-1} - f(1-s)s^{2H-1} \\
&= s^{2H-1}(1-s)^{2H-1}\Big(f(s)s^{1-2H} - f(1-s)(1-s)^{1-2H}\Big),
\end{aligned}$$

where in the second to last equality we used the identity $f(s) + f(1-s) = 1$. We can now represent $G_\alpha(s)$ as

$$G_\alpha(s) = s^{2H-1}(1-s)^{2H-1}\Big(A(s) - A(1-s)\Big), \qquad A(s) = f(s)s^{1-2H}.$$

Similarly, but using the identity $f(s) - f(1-s) = s^{2H} - (1-s)^{2H}$, we obtain a representation of $G_0(s)$

$$\begin{aligned}
G_0(s) &= \Big(s^{2H-1} + (1-s)^{2H-1}\Big)s^{2H} - f(s)s^{2H-1} \\
&= \Big(s^{2H-1} + (1-s)^{2H-1}\Big)s^{2H} - \Big(f(1-s) + s^{2H} - (1-s)^{2H}\Big)s^{2H-1} \\
&= -f(1-s)s^{2H-1} + \Big((1-s)^{2H-1}s^{2H} + (1-s)^{2H}s^{2H-1}\Big) \\
&= s^{2H-1}(1-s)^{2H-1}\Big(-A(1-s) + 1\Big).
\end{aligned}$$

We can now rewrite (2.122) as

$$D'_\alpha(s) = \widetilde{D}_\alpha(s)\, \widetilde{G}_\alpha(s), \tag{2.123}$$

with

$$\widetilde{G}_\alpha(s) \coloneqq \alpha\Big(A(s) - A(1-s)\Big) - A(1-s) + 1, \tag{2.124}$$

$$\widetilde{D}_\alpha(s) \coloneqq \frac{2H(\alpha + f(s))s^{2H-1}(1-s)^{2H-1}}{(s^{2H} - f^2(s))^2}. \tag{2.125}$$

We claim now that the function $\widetilde{G}_\alpha(s)$ is increasing. Provided that this is true, we immediately obtain both existence and uniqueness of the optimal point s_*, as well as positivity of the second derivative at this point. Indeed,

$$D''_\alpha(s_*) = \widetilde{D}'_\alpha(s_*)\, \underbrace{\widetilde{G}_\alpha(s_*)}_{=0} + \underbrace{\widetilde{D}_\alpha(s_*)}_{>0}\, \underbrace{\widetilde{G}'_\alpha(s_*)}_{>0} > 0. \tag{2.126}$$

To prove this claim it clearly suffices to show that $A(s)$ is increasing.

We have,

$$A'(s) = f'(s)s^{1-2H} + (1-2H)f(s)s^{-2H}$$

and the positivity of $A'(s)$ is equivalent to that of

$$sf'(s) + (1-2H)f(s).$$

In case $H \le 1/2$, the inequality

$$sf'(s) + (1-2H)f(s) > 0$$

is clear, since $f'(s) > 0$. If $H > 1/2$, we use the Bernoulli inequality

$$(1-s)^{2H-1} \le 1 - (2H-1)s, \quad s \in [0,1],$$

which gives

$$\begin{aligned} sf'(s) + (1-2H)f(s) &= (1-2H) + s^{2H} - (1-s)^{2H} + 2H(1-s)^{2H-1} \\ &\ge (1-2H) + s^{2H} - (1-s)\Big(1-(2H-1)s\Big) + 2H(1-s)^{2H-1} \\ &= s^{2H} - (2H-1)s^2 + 2H\Big((1-s)^{2H-1} - (1-s)\Big) \\ &\ge s^{2H} - (2H-1)s^2 = s^2\Big(s^{2H-2} - 2H + 1\Big) \\ &\ge 2s^2\Big(1-H\Big) > 0. \end{aligned}$$

As a corollary of the above, we obtain $\alpha\, G_\alpha(s_*) + G_0(s_*) = 0$ or

$$\begin{aligned} \alpha &\left[r(s_*,1)\Big(s_*^{2H-1} + (1-s_*)^{2H-1}\Big) - s_*^{2H-1} \right] \\ &\qquad + \left[s_*^{2H}\Big(s_*^{2H-1} + (1-s_*)^{2H-1}\Big) - r(s_*,1)s_*^{2H-1} \right] = 0, \end{aligned} \tag{2.127}$$

which will be useful for us in Lemma 2.4.

We have thus shown that the function

$$\sigma_{\boldsymbol{b}}^{-2}(t,s) = \frac{a^2 s^{2H} + 2abr(t,s) + b^{2H}t^2}{(ts)^{2H} - r^2(t,s)}$$

posesses a unique minimum in the lower triangle. By (2.121), we see that this point is given by

$$t_* = Ts_*,$$

where s_* is the minimizer of D. Moreover, we have by (2.123)

$$\frac{\partial \sigma_{\boldsymbol{b}}^{-2}}{\partial s}(T, t_*) = \frac{a^2}{T^{2H}} \widetilde{D}_{b/a}(s_*) \underbrace{\widetilde{G}_{b/a}(s_*)}_{=0} = 0$$

and by (2.126) we obtain

$$\kappa_2 := \frac{\partial^2 \sigma_{\boldsymbol{b}}^{-2}}{\partial s^2}(T, t_*) = \frac{a^2}{T^{2H}}\, D''_{b/a}(s_*) = \frac{a^2}{T^{2H}}\, \widetilde{D}_{b/a}(s_*)\widetilde{G}'_{b/a}(s_*) > 0$$

Similarly to (2.121), let us rewrite $\sigma_{\boldsymbol{b}}^{-2}$ as follows:

$$\sigma_{\boldsymbol{b}}^{-2}(t, s) = \frac{a^2}{t^{2H}}\, D_{b/a}\left(\frac{s}{t}\right)$$

implying

$$-\kappa_1 := \frac{\partial \sigma_{\boldsymbol{b}}^{-2}}{\partial t}(T, t_*) = -\frac{2Ha^2}{T^{2H+1}}\, D_{b/a}(s_*) + \frac{a^2}{T^{2H}}\left[\frac{-s_*}{T^2}\right] \underbrace{D'_{b/a}(s_*)}_{=0} < 0.$$

Finally, we have

$$\sigma_{\boldsymbol{b}}^{-2}(\boldsymbol{t}_*) - \sigma_{\boldsymbol{b}}^{-2}(\boldsymbol{t}_* - \boldsymbol{\tau}) \sim -\kappa_1\, \tau_1 - \kappa_2\, \tau_2^2.$$

□

2.6.10 Double crossing for fBm: matrix expansions

Proof of Lemma 2.4. Let $\boldsymbol{t}_* = (T, t_*)$ be a point in $[0, T]^2$ minimizing the generalized variance $\sigma_{a,b}^{-2}(\boldsymbol{t})$.

Recall that

$$\Sigma(\boldsymbol{t}_*) = \begin{pmatrix} T^{2H} & -r(T, t_*) \\ -r(T, t_*) & t_*^{2H} \end{pmatrix}, \qquad R(\boldsymbol{t}, \boldsymbol{s}) = \begin{pmatrix} r(t_1, s_1) & -r(t_1, s_2) \\ -r(t_2, s_1) & r(t_2, s_2) \end{pmatrix},$$

where

$$r(t, s) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t - s|^{2H}\right).$$

We have

$$\Sigma(\boldsymbol{t}_*) - R(\boldsymbol{t}_* + \boldsymbol{t}, \boldsymbol{t}_* + \boldsymbol{s}) = \begin{pmatrix} T^{2H} - r(T + t_1, T + s_1) & -r(T, t_*) + r(T + t_1, t_* + s_2) \\ -r(t_*, T) + r(t_* + t_2, T + s_1) & t_*^{2H} - r(t_* + t_2, t_* + s_2) \end{pmatrix}.$$

For the top left cell, we have:

$$T^{2H} - r(T + t_1, T + s_1) = T^{2H} - \frac{1}{2}\left((T + t_1)^{2H} + (T + s_1)^{2H} - |t_1 - s_1|^{2H}\right).$$

Here is the expression for the top right cell:

$$r(T + t_1, t_* + s_2) - r(T, t_*) = \frac{1}{2}\left((T + t_1)^{2H} - T^{2H}\right) + \frac{1}{2}\left((t_* + s_2)^{2H} - t_*^{2H}\right) - \frac{1}{2}\left(|T - t_* + t_1 - s_2|^{2H} - |T - t_*|^{2H}\right)$$

and similarly for the remaining two. Let us compute the first order coefficients of different contributions. Jumping ahead, we will be giving these coefficients names corresponding to their roles within Assumption A2. Recall that the first index i in $A_{i,j}$ corresponds to the order (first or second) of the contribution, while the second indicates the variable t_j. The coefficients of the corresponding $\boldsymbol{s}$-terms can be expressed as transpositions of these.

First, the only two terms which depend on the difference are the following:

$$|t_1 - s_1|^{2H}: \quad A_{5,1} := \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad |t_2 - s_2|^{2H}: \quad A_{5,2} := \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \alpha_1 = \alpha_2 = 2H.$$

Next, we proceed to the power-type contributions of the leading order:

$$t_1\colon \quad A_{2,1} := H\begin{pmatrix} -T^{2H-1} & T^{2H-1} - |T - t_*|^{2H-1} \\ 0 & 0 \end{pmatrix},$$

$$t_2\colon \quad A_{1,2} := H\begin{pmatrix} 0 & 0 \\ t_*^{2H-1} + |T - t_*|^{2H-1} & -t_*^{2H-1} \end{pmatrix}.$$

We will show below that

$$\boldsymbol{w}^\top A_{2,1}\, \boldsymbol{w} > 0 \quad \text{and} \quad A_{1,2}\, \boldsymbol{w}(\boldsymbol{t}) \sim 0, \tag{2.128}$$

whence the names $A_{2,1}$ and $A_{1,2}$. It also explains why we do not need to compute the second order in t_1. However, we do need to find the second order in the second coordinate:

$$t_2^2\colon \quad A_{2,2} := H\left(H - \frac{1}{2}\right)\begin{pmatrix} 0 & 0 \\ t_*^{2H-2} + |T - t_*|^{2H-2} & -t_*^{2H-2} \end{pmatrix}, \tag{2.129}$$

and the coefficient $A_{6,2,2}$ of $t_2\, s_2$ is zero. To show (2.128), we need the inverse of Σ

$$\Sigma^{-1}(\boldsymbol{t}) = \frac{1}{t_1^{2H} t_2^{2H} - r^2(t_1,t_2)}\begin{pmatrix} t_2^{2H} & r(t_1,t_2) \\ r(t_1,t_2) & t_1^{2H} \end{pmatrix}$$

and its action on the vector $\boldsymbol{w}(\boldsymbol{t})$:

$$\boldsymbol{w}(\boldsymbol{t}) = \Sigma^{-1}(\boldsymbol{t})\, \boldsymbol{b} = \frac{1}{t_1^{2H} t_2^{2H} - r^2(t_1,t_2)}\begin{pmatrix} t_2^{2H}\, a + r(t_1,t_2)\, b \\ r(t_1,t_2)\, a + t_1^{2H}\, b \end{pmatrix}.$$

Using the following identity (2.127) from Lemma 2.3

$$b\left[r(s_*,1)\left(s_*^{2H-1} + (1-s_*)^{2H-1}\right) - s_*^{2H-1}\right] + a\left[s_*^{2H}\left(s_*^{2H-1} + (1-s_*)^{2H-1}\right) - r(s_*,1)s_*^{2H-1}\right] = 0$$

we can show that Assumptions A2.3 to A2.5 are satisfied with

$$A_{1,2}\, \boldsymbol{w}(\boldsymbol{t}) \sim 0, \quad \boldsymbol{w}^\top A_{2,1}\, \boldsymbol{w} > 0, \quad \boldsymbol{w}^\top A_{5,2}\, \boldsymbol{w} > 0, \quad \beta_1 = 1, \quad \beta_2 = 2, \quad \mathcal{F} = \{2\},$$

which gives

$$\Sigma - R(\boldsymbol{t}_* + \boldsymbol{t}, \boldsymbol{t}_* + \boldsymbol{s}) = \left[A_{2,1}\, t_1 + A_{1,2}\, t_2 + A_{2,2}\, t_2^2\right] + \left[A_{2,1}^\top\, s_1 + A_{1,2}^\top\, s_2 + A_{2,2}^\top\, s_2^2\right]$$
$$+ A_{5,1}\, |t_1 - s_1|^{2H} + A_{5,2}\, |t_2 - s_2|^{2H} + o\left(\sum_{i=1}^{n}\left[t_1 + t_2^2 + s_1 + s_2^2\right]\right),$$

which verifies A2.1 and A2.2. Finally, $\mathcal{F}\times\mathcal{F}$ contains one element $(2,2)$, and by (2.129) we have that

$$A_{6,2,2} + A_{1,2}\Sigma^{-1}A_{1,2}^\top = A_{1,2}\Sigma^{-1}A_{1,2} = C_2\, C_2^\top \quad \text{with} \quad C_2 = A_{1,2}\, \Sigma^{-1/2},$$

so A2.6 is satisfied. □

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Chapter 3

A matrix-valued Schoenberg's problem and its applications

In this chapter we present a criterion for positive definiteness of the matrix-valued function

$$f(t) := \exp\Big(-|t|^{\alpha}\Big[B^{+} + B^{-}\,\mathrm{sign}(t)\Big]\Big),$$

where $\alpha \in (0,2]$ and $B^{\pm}$ are real symmetric and antisymmetric $d \times d$ matrices. We also find a criterion for positive definiteness of its multidimensional generalization

$$f(t) := \exp\left(-\int_{\mathbb{S}^{d-1}} \left|\boldsymbol{t}^{\top}\boldsymbol{s}\right|^{\alpha} \left[B^{+} + B^{-}\,\mathrm{sign}(\boldsymbol{t}^{\top}\boldsymbol{s})\right] d\Lambda(\boldsymbol{s})\right),$$

where Λ is a finite measure on the unit sphere $\mathbb{S}^{d-1} \subset \mathbb{R}^d$ under a more restrictive assumption that $B^{\pm}$ commute and are normal. The associated stationary Gaussian random field may be viewed as as a generalization of the univariate fractional Ornstein-Uhlenbeck process. This generalization turns out to be particularly useful for the asymptotic analysis of $\mathbb{R}^d$-valued Gaussian random fields. Another possible application of these findings may concern variogram modelling and general stationary time series analysis.

Pavel Ievlev and Svyatoslav Novikov, *A matrix-valued Schoenberg's problem and its applications*, Electronic Communications in Probability **28** (2023), Paper No. 48, 12. MR4684061

3.1 Introduction

In 1938, Schoenberg [1] posed the problem of determining for which numbers $\alpha > 0$ and norms $\|\cdot\|$ on $\mathbb{R}^n$ the function $\boldsymbol{t} \mapsto \exp(-\|\boldsymbol{t}\|^\alpha)$ is positive-definite. The complete solution to this problem has been given in 1992 by Koldobsky [2] for the case where $\|\cdot\| = \|\cdot\|_q$. It is clear that if $(\alpha, \|\cdot\|)$ is a pair for which this function is positive definite, then for any $B \geq 0$, the function $\boldsymbol{t} \mapsto \exp(-B\|\boldsymbol{t}\|^\alpha)$ is also positive definite. However, the question arises whether we can take $B \in \mathbb{C}$ instead of $B \geq 0$? The application of Bochner's theorem shows that the answer is negative. Nonetheless, we can modify this function to make its Fourier transform real. Let $n = 1$, let $\|\cdot\|$ be the absolute value and consider the following function

$$t \mapsto \exp(-B|t|^\alpha)\, \mathbb{1}_{t\geq 0} + \exp\left(-\overline{B}|t|^\alpha\right) \mathbb{1}_{t<0}, \qquad t \in \mathbb{R}. \tag{3.1}$$

This family of functions, parameterized by α and B, is of great importance in the theory of stable distributions. This theory provides us with the following answer [3, Remark (7.26)]: (3.1) is positive definite if and only if

$$\alpha \in (0, 2], \qquad \operatorname{Re} B \geq 0 \quad \text{and} \quad |\operatorname{Im} B| \leq \operatorname{Re} B \cdot \left|\tan\left(\frac{\pi\alpha}{2}\right)\right|. \tag{3.2}$$

Motivated by applications in the theory of Gaussian processes (more on that below), we aim to extend this result to the case where B is a $d \times d$ matrix. Specifically, we investigate the necessary and sufficient conditions for positive-definiteness of the matrix-valued function f defined analogously to (3.1) by

$$f(t) := \exp(-B|t|^\alpha)\, \mathbb{1}_{t\geq 0} + \exp\left(-B^\top |t|^\alpha\right) \mathbb{1}_{t<0}, \qquad t \in \mathbb{R},$$

with $\alpha \in (0, 2]$ and B a real $d \times d$ matrix. We will mostly use the following representation of f:

$$f(t) = \exp\left(-|t|^\alpha \left[B^+ + B^- \operatorname{sign}(t)\right]\right), \quad \text{where} \quad B^\pm := \frac{B \pm B^\top}{2}. \tag{3.3}$$

The counterpart of the condition (3.2) in this case is

$$\widetilde{B} := B^+ \sin\left(\frac{\pi\alpha}{2}\right) - iB^- \cos\left(\frac{\pi\alpha}{2}\right) \trianglerighteq 0. \tag{3.4}$$

Here $\trianglerighteq$ denotes positive definiteness. As it turns out, for $\alpha \in [1, 2)$ condition (3.4) is both necessary and sufficient for positive definiteness of f, whereas for $\alpha \in (0, 1)$ it is necessary, but not sufficient. If $\alpha = 2$, we need to assume that $B^+ \trianglerighteq 0$. This is the subject of Theorem 3.3.

In Theorem 3.2 we present a multivariate extension of this result under a restrictive assumption of B being normal (unitarily diagonalizable). More specifically, we use the theory of multivariate stable laws (see e.g. [4, Chapter 2]) to show that under the same assumption (3.4) for every finite measure Λ on the unit sphere $\mathbb{S}^{d-1} \subset \mathbb{R}^d$ the function

$$f(\boldsymbol{t}) = \exp\left(-\int_{\mathbb{S}^{d-1}} \left|\boldsymbol{t}^\top \boldsymbol{s}\right|^\alpha \left[B^+ + B^- \operatorname{sign}(\boldsymbol{t}^\top \boldsymbol{s})\right] d\Lambda(\boldsymbol{s})\right), \qquad \boldsymbol{t} \in \mathbb{R}^d$$

is positive definite.

Surprisingly, the condition (3.4) arises also from the study of operator fractional Brownian motions [5, Remark 8]. As it turns out, this is a necessary and sufficient condition for positive definiteness of the following matrix-valued function

$$R(t, s) = B|t|^\alpha + B^\top |s|^\alpha - B|t - s|^\alpha \quad \text{for} \quad t > s$$

and satisfying $R^\top(t,s) = R(s,t)$. This class of multivariate fBm's is not only interesting in its own right, but is also essential in the theory of multivariate Gaussian extremes [6]. The occurrence of the same condition in both problems is not a coincidence. In fact, this observation leads to significant simplifications in the study of multivariate Gaussian extremes, which will be the subject of our upcoming paper on the extremes of locally-stationary $\mathbb{R}^d$-valued random fields. More specifically, the classical Pickands-Piterbarg approach to the asymptotical analysis of high exceedance probabilities of a non-stationary Gaussian process $X(t)$, $t \in \mathbb{R}$ heavily relies on the possibility to find a pair of stationary processes $Y_\pm(t)$, $t \in \mathbb{R}$, which stochastically dominate X from above and from below and are close to X on a given short interval. If X satisfies some weak assumptions, we can take $Y_\pm$ to be the processes associated to the covariance functions $e^{-B_\pm |t|^\alpha}$, $t \in \mathbb{R}$ with specially chosen $B_\pm$, and apply the Slepian inequality. In the case of $\mathbb{R}^d$-valued processes, the same approach with the Gordon inequality instead of Slepian's prompts the consideration of the process associated to (3.3). It remains, however, to show that (3.3) is a covariance function, which is exactly what we study in this paper.

The importance of our results is twofold:

1. they can be used to construct valid covariance functions of $\mathbb{R}^d$-valued Gaussian random fields, and

2. they can be used for cross-variogram and pseudo-variogram modelling, which is important for statistical applications, see [7].

More specifically, positive definiteness of f implies that the following function

$$t \mapsto I - \frac{1}{2}\left[\exp\left(-|t|^\alpha\Big[B^+ + B^- \operatorname{sign}(t)\Big]\right) + \exp\left(-|t|^\alpha\Big[B^+ - B^- \operatorname{sign}(t)\Big]\right)\right], \qquad t \in \mathbb{R},$$

with I the identity matrix is a cross-variogram and

$$t \mapsto J - \exp\left(-|t|^\alpha\Big[B^+ + B^- \operatorname{sign}(t)\Big]\right), \qquad t \in \mathbb{R},$$

with $J_{ij} = 1$ (matrix of all ones) is a pseudo-variogram. Under the assumptions of Theorem 3.2, the same is true for the functions

$$\begin{aligned} \boldsymbol{t} \mapsto I - \frac{1}{2}\Bigg[&\exp\left(-\int_{\mathbb{S}^{d-1}} \left|\boldsymbol{t}^\top \boldsymbol{s}\right|^\alpha \left[B^+ + B^- \operatorname{sign}(\boldsymbol{t}^\top \boldsymbol{s})\right] d\Lambda(\boldsymbol{s})\right) \\ &+ \exp\left(-\int_{\mathbb{S}^{d-1}} \left|\boldsymbol{t}^\top \boldsymbol{s}\right|^\alpha \left[B^+ - B^- \operatorname{sign}(\boldsymbol{t}^\top \boldsymbol{s})\right] d\Lambda(\boldsymbol{s})\right)\Bigg], \qquad \boldsymbol{t} \in \mathbb{R}^d, \end{aligned}$$

and

$$\boldsymbol{t} \mapsto J - \exp\left(-\int_{\mathbb{S}^{d-1}} \left|\boldsymbol{t}^\top \boldsymbol{s}\right|^\alpha \left[B^+ + B^- \operatorname{sign}(\boldsymbol{t}^\top \boldsymbol{s})\right] d\Lambda(\boldsymbol{s})\right), \qquad \boldsymbol{t} \in \mathbb{R}^d.$$

Finally, let us briefly mention that there are two close relatives of the family of processes corresponding to f: the operator fractional Ornstein-Uhlenbeck process $\boldsymbol{X}(t)$, $t \in \mathbb{R}$ from [6, Section 3.1], associated to the covariance $\operatorname{Cov}(\boldsymbol{X}(t), \boldsymbol{X}(s)) = \exp(-|t-s|^\alpha)$, where α is a symmetric $d \times d$ matrix with eigenvalues belonging to $(0, 2]$, and the multivariate Ornstein-Uhlenbeck process, defined as a solution of a certain stochastic differential equation driven by a Brownian motion. The covariance of the latter is given by $\int_0^t e^{-A(t-s)} B e^{-A(t-s)}\, ds$, where A and B are real $d \times d$ matrices satisfying some additional assumptions. See, for example, [8, 9, 10].

3.1.1 Brief organization of the paper

Section 3.2 contains our main results. It begins with a simplified version of the main theorem along with its proof, after which we formulate an extension of this simplified version to multidimensional time. The main result of this contribution is Theorem 3.3. In Section 3.3 we reproduce for reader's convenience three known theorems (Operator-valued Bochner's Theorem, Bernstein's theorem on completely monotone functions and the Canonical representations of univariate and multivariate stable laws). The proof of the main theorem is presented in Section 3.4. More technical results are relegated to the Appendix.

3.1.2 Notation.

Throughout the paper, we use the term "positive-definite" to refer to nonnegative-definite functions. To emphasize the case when the inequality is strict, we use the expression "strictly positive-definite."

If f is a matrix-valued function, we write $f \succeq 0$ to indicate that f is a positive-definite function in the following sense: $f^\top(t) = f(-t)$ and

$$\sum_{k,m=1}^{n} \boldsymbol{z}_k^* \, f(t_k - t_m) \, \boldsymbol{z}_m \geq 0, \qquad \forall \, \{\boldsymbol{z}_k\}_{k=1,\dots,n} \subset \mathbb{C}^d, \; \{t_k\}_{k=1,\dots,n} \subset \mathbb{R}.$$

We will utilize the same notation $f \succeq 0$ is f is a complex-valued positive-definite function. Occasionally, we write the sign $\succ$ to indicate that the positive definiteness is strict.

We also write $A \trianglerighteq 0$ for a matrix A to indicate that A is a positive-definite matrix in the usual sense, namely,

$$A = A^* \quad \text{and} \quad \boldsymbol{z}^* A \, \boldsymbol{z} \geq 0, \qquad \forall \, \boldsymbol{z} \in \mathbb{C}^d.$$

The corresponding strict version will be denoted by $\triangleright$.

Note that we will write "$f \trianglerighteq 0$" to say that a matrix-valued function f is positive-definite *as a matrix*, rather than as a function. The difference between these two notions is crucial for the matrix-valued Bochner's Theorem 3.4.

The Fourier transform of a function f is defined by

$$\mathcal{F}[f](\xi) = \int_{\mathbb{R}} e^{i\xi x} f(x) \, dx \, .$$

The application of $\mathcal{F}$ to matrix-valued functions is performed component-wise.

3.2 Main results

In order to provide the reader with an intuitive understanding of the main result, we begin by presenting a preliminary, simpler version of the theorem that serves as a warm-up example.

More specifically, assume that B is normal, i.e., there exists a diagonal matrix D and an unitary matrix P such that

$$B = P^* D P. \tag{3.5}$$

The positive definiteness of f is therefore equivalent to that of

$$g(t) := \exp\Big(-|t|^\alpha \Big[D^+ + D^- \operatorname{sign}(t)\Big]\Big) \, .$$

Since D^- is anti-Hermitian, its elements are purely imaginary. We denote them by $-i\lambda_k^- := (D^-)_{kk}$, where $\lambda_k^- \in \mathbb{R}$. Similarly, the k-th element of D^+ is denoted by $\lambda_k^+ := (D^+)_{kk}$.

Assume further that the matrix B satisfies (3.4). As mentioned in the Introduction, this condition turns out to be necessary for positive-definiteness of f. See Section 3.4.1 for the proof.

By definition of positive definiteness, $g \succeq 0$ if and only if

$$\sum_{i,j} \boldsymbol{z}_i^\top \exp\left(-|t_i - t_j|^\alpha \Big[D^+ + D^- \operatorname{sign}(t_i - t_j)\Big]\right) \boldsymbol{z}_j$$

$$= \sum_k \sum_{i,j} z_{ik} \exp\left(-|t_i - t_j|^\alpha \Big[\lambda_k^+ - i\lambda_k^- \operatorname{sign}(t_i - t_j)\Big]\right) z_{jk}$$

is non-negative for all possible choices of $\boldsymbol{z}_k \in \mathbb{C}^n$ and $t_i \in \mathbb{R}$. We will show a stronger claim: for all k, the scalar-valued function

$$g_k(t) := \exp\left(-|t|^\alpha \Big[\lambda_k^+ - i\lambda_k^- \operatorname{sign}(t)\Big]\right)$$

is positive-definite. This function is known as the characteristic function of the α-stable law, which by the well-known canonical representation theorem (see Theorem 3.6 and Remark 3.2) is positive definite if and only if

$$\lambda_k^+ \geq 0 \quad \text{and} \quad \left|\lambda_k^-\right| \leq \lambda_k^+ \left|\tan\left(\frac{\pi\alpha}{2}\right)\right|. \tag{3.6}$$

If $\alpha \neq 2$, then the assumption $\widetilde{B} \trianglerighteq 0$ implies that these conditions are met. Indeed, if $\widetilde{B} \trianglerighteq 0$, then

$$\lambda_k^+ \sin\left(\frac{\pi\alpha}{2}\right) + \lambda_k^- \cos\left(\frac{\pi\alpha}{2}\right) \geq 0 \quad \text{and} \quad \lambda_k^+ \sin\left(\frac{\pi\alpha}{2}\right) - \lambda_k^- \cos\left(\frac{\pi\alpha}{2}\right) \geq 0,$$

from which the inequalities (3.6) easily follow. If $B \trianglerighteq 0$, then the inequalities (3.6) are also satisfied. We have thus proven the following result.

Theorem 3.1. *If $\alpha \in (0,2)$ and B is a real $d \times d$ normal matrix satisfying* (3.4)*, then the function defined in* (3.3) *is positive-definite. If $\alpha = 2$ and B is a real $d \times d$ matrix satisfying $B \trianglerighteq 0$, then the function defined in* (3.3) *is positive-definite.*

By the same proof as above with the use of Theorem 3.7 instead of Theorem 3.6, we obtain the following generalization.

Theorem 3.2 (Multivariate parameter extension of the previous result)**.** *Let $\alpha \in (0,2]$, B is a real $d \times d$ normal matrix satisfying* (3.4) *or $B^+ \trianglerighteq 0$ if $\alpha = 2$, and Λ is a finite measure on the unit sphere $\mathbb{S}^{d-1} \subset \mathbb{R}^d$, then the function defined by*

$$f(\boldsymbol{t}) := \exp\left(-\int_{\mathbb{S}^{d-1}} \left|\boldsymbol{t}^\top \boldsymbol{s}\right|^\alpha \left[B_+ + B_- \operatorname{sign}(\boldsymbol{t}^\top \boldsymbol{s})\right] d\Lambda(\boldsymbol{t})\right)$$

is positive definite.

We now proceed to the statement of the general theorem. For $p > 0$ and a matrix A with spectrum in $\mathbb{C} \setminus (-\infty, 0]$, define

$$A^p := \exp\left(p(A - I)\int_0^1 [s(A-I)+I]^{-1}\, ds\right).$$

Theorem 3.3. *Let B be a real $d \times d$ matrix. If the function f defined in* (3.3) *is positive-definite, then the conditions* (3.4) *and $B^+ \trianglerighteq 0$ are satisfied. If, on the other hand, the condition* (3.4) *is satisfied, then*

- *If $\alpha \in (0,1)$ and B is invertible, then f is positive-definite if and only if B additionally satisfies*

$$B^{1/\alpha} + B^{1/\alpha,\top} \trianglerighteq 0. \tag{3.7}$$

- *If $\alpha \in [1,2)$, then f is positive definite.*
- *If $\alpha = 2$ and $B^+ \trianglerighteq 0$, then f is positive definite.*

Remark 3.1. *If B is normal, the condition* (3.7) *follows from* (3.4)*. See Section 3.4.2 for the proof.*

3.3 Auxiliary results

3.3.1 Operator-valued Bochner's theorem

The following version of Bochner's theorem is taken from [11, Theorem III.3].

Theorem 3.4 (Operator-valued Bochner Theorem, Neeb 1998)**.** *Let G be a locally compact abelian group, $\widehat{G}$ its character group, and $\mathcal{H}$ a Hilbert space. Then an ultraweakly continuous function $K\colon G \to B(\mathcal{H})$, where $B(\mathcal{H})$ is the set of bounded operators on $\mathcal{H}$, is positive definite if and only if there exists a finite* $\mathrm{Herm}^+(\mathcal{H})$*-valued measure μ on $\widehat{G}$ such that*

$$K(g) = \int_{\widehat{G}} \chi(g)\, d\mu(\chi)\,.$$

Here $\mathrm{Herm}^+(\mathcal{H})$ *is the cone of bounded positive-definite Hermitian operators on $\mathcal{H}$. The Radon measure μ is uniquely determined by K.*

We are interested in the particular case of this lemma where $G = \mathbb{R}$ and $\mathcal{H} = \mathbb{R}^d$.[1]

Corollary 3.1 (Matrix-valued Bochner Theorem on $\mathbb{R}$)**.** *A continuous matrix-valued function $f\colon \mathbb{R} \to \mathbb{C}^{d\times d}$ is positive definite $f \succeq 0$ if and only if there exists a matrix-valued measure $\mu \trianglerighteq 0$ on $\mathbb{R}$ such that*

$$f(t) = \int_{\mathbb{R}} e^{-i\xi t}\, d\mu(\xi)\,.$$

3.3.2 Bernstein's theorem

An infinitely differentiable function $f\colon (0,\infty) \to \mathbb{R}_+$ is said to be completely monotone if for any non-negative integer $n \geq 0$ holds

$$(-1)^n \frac{d^n f}{dt^n}(t) \geq 0, \qquad t > 0.$$

The following version of the celebrated Bernstein's theorem on completely monotone functions is taken from [3, Theorem A.3.6].

[1] See also [12, Theorem 2.10].

Theorem 3.5 (Bernstein 1928). *A real-valued function f is completely monotone if and only if there exists a measure μ on $(0,\infty)$ such that*

$$f(t) = \int_0^\infty e^{-ut}\, d\mu(u)\,.$$

This measure is finite if $\lim_{t\downarrow 0} f(t) < \infty$.

In particular, we will use this theorem with $\exp(-x^\alpha)$ if $\alpha \in (0,1)$, for which μ is finite, and $\exp(-x^{1/\alpha})\, x^{1/\alpha-1}/\alpha$ if $\alpha \in (1,2)$, for which μ is infinite.

3.3.3 Canonical representation of stable laws

Two basic sources on α-stable laws are the monograph by Steutel & van Harn [3] and the monograph by Uchaikin & Zolotarev [13]. We will need the following result, which is taken from [3, Theorem 7.11].

Theorem 3.6 (Canonical representation of α-stable laws). *For $\alpha \in (0,2]\setminus\{1\}$, a $\mathbb{C}$-valued function f on $\mathbb{R}^d$ is the characteristic function of a centered non-degenerate stable distribution with exponent α if and only if it is of the form*

$$f(t) = \exp\Big(-|t|^\alpha \Big[\lambda - i\theta\,\mathrm{sign}(t)\Big]\Big)\,, \tag{3.8}$$

where $\lambda > 0$ and θ satisfies

$$|\theta| \le \lambda \left|\tan\left(\frac{\pi\alpha}{2}\right)\right|. \tag{3.9}$$

Remark 3.2. *As remarked in [3, near formula (7.26)], it follows from this theorem combined with some simple considerations that the function defined in* (3.8) *with $\lambda > 0$ is positive definite* if and only if *the condition* (3.9) *is satisfied.*

If $\lambda = 0$ and $\alpha \neq 1$, the function (3.8) *is positive-definite if and only if $\theta = 0$. In case $\alpha = 1$,* $\exp(i\theta t)$ *is positive definite for any $\theta \in \mathbb{R}$.*

The following extension of the previous theorem immediately follows from [4, Theorem 2.3.1].

Theorem 3.7 (Canonical representation of multivariate α-stable laws). *For $\alpha \in (0,2]\setminus\{1\}$, a $\mathbb{C}$-valued function f on $\mathbb{R}$ is the characteristic function of a centered non-degenerate stable random vector with exponent α if and only if there exists a finite measure Λ on the unit sphere $\mathbb{S}^{d-1} \subset \mathbb{R}^d$ such that*

$$f(\boldsymbol{t}) = \exp\left(-\int_{\mathbb{S}^{d-1}} \left|\boldsymbol{t}^\top \boldsymbol{s}\right|^\alpha \Big[\lambda - i\theta\,\mathrm{sign}(\boldsymbol{t}^\top \boldsymbol{s})\Big]\, d\Lambda(\boldsymbol{s})\right),$$

where $\lambda > 0$ and θ satisfies (3.9).

Remark 3.3. *As above, if $\alpha = 1$, there are no restrictions on θ.*

3.4 Proofs

3.4.1 Necessary condition

Proof of necessity in Theorem 3.3. Suppose that $f \succeq 0$. For $\boldsymbol{z} \in \mathbb{C}^n$, define

$$f_{\boldsymbol{z}}(t) := \boldsymbol{z}^* f(t)\, \boldsymbol{z}.$$

It is easy to see that these functions satisfy $f_{\boldsymbol{z}} \succeq 0$.[2] Note that

$$\left[\frac{1}{\boldsymbol{z}^*\boldsymbol{z}} f_{\boldsymbol{z}}\left(\left(\frac{\boldsymbol{z}^*\boldsymbol{z}}{n}\right)^{1/\alpha} t\right)\right]^n = \left[I - \frac{|t|^\alpha}{n}\boldsymbol{z}^*\Big[B^+ + B^- \operatorname{sign}(t)\Big]\boldsymbol{z} + O\left(\frac{1}{n^2}\right)\right]^n \xrightarrow[n\to\infty]{} g_{\boldsymbol{z}}(t)$$

with

$$g_{\boldsymbol{z}}(t) = \exp\left(-|t|^\alpha\, \boldsymbol{z}^*\Big[B^+ + B^- \operatorname{sign}(t)\Big]\boldsymbol{z}\right). \tag{3.10}$$

Since positive-definiteness of scalar-valued functions is preserved under stretching, taking powers and taking limits, we have $g_{\boldsymbol{z}} \succeq 0$.

Since B^- is real antisymmetric, iB^- is Hermitian and therefore $\boldsymbol{z}^* iB^-\boldsymbol{z} \in \mathbb{R}$. By Theorem 3.6 and Remark 3.2, this function is positive-definite if and only if

$$\boldsymbol{z}^*B^+\boldsymbol{z} \geq 0, \qquad \left|\boldsymbol{z}^*iB^-\boldsymbol{z}\right| \leq \boldsymbol{z}^*B^+\boldsymbol{z}\left|\tan\left(\frac{\pi\alpha}{2}\right)\right|.$$

Multiplying both sides by $|\cos(\pi\alpha/2)|$ and noting that $\sin(\pi\alpha/2) \geq 0$ for $\alpha \in (0,2]$, we find that

$$\pm\boldsymbol{z}^*iB^-\boldsymbol{z}\,\cos\left(\frac{\pi\alpha}{2}\right) \leq \boldsymbol{z}^*B^+\boldsymbol{z}\,\sin\left(\frac{\pi\alpha}{2}\right),$$

which is equivalent to

$$\widetilde{B} \coloneqq B^+\sin\left(\frac{\pi\alpha}{2}\right) - iB^-\cos\left(\frac{\pi\alpha}{2}\right) \succeq 0$$

because B is real. □

3.4.2 Alternative form of the B condition and the eigenvalues of B

Note that

$$i^{\alpha-1} = \sin\left(\frac{\pi\alpha}{2}\right) - i\cos\left(\frac{\pi\alpha}{2}\right).$$

Hence, $\widetilde{B} \trianglerighteq 0$ is equivalent to

$$i^{\alpha-1}B + i^{1-\alpha}B^\top \trianglerighteq 0.$$

Therefore, if λ is an eigenvalue of B, it satisfies

$$\operatorname{Re}\, i^{\alpha-1}\lambda \geq 0 \quad \text{and} \quad \operatorname{Re}\, i^{1-\alpha}\lambda \geq 0$$

because B is real. Rewriting both in terms of their arguments, we obtain

$$\pm\frac{(\alpha-1)\pi}{2} + \arg\lambda \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

It follows that

$$\arg\lambda \in \left[-\frac{\pi\alpha}{2}, \frac{\pi\alpha}{2}\right]$$

and therefore

$$\operatorname{Re}\,\lambda^{1/\alpha} \geq 0. \tag{3.11}$$

If B is normal, (3.11) implies that

$$B^{1/\alpha} + B^{1/\alpha,\top} = P^\top\left[D^{1/\alpha} + D^{*,1/\alpha}\right]P \trianglerighteq 0.$$

and the condition (3.7) is satisfied.

[2] Although we will not use it, we want to mention that positive definiteness of scalar-valued projections of a matrix-valued function $f(t,s)$ is in fact equivalent to the positive-definiteness of f itself if $f(t,s) = f(t-s)$. This was originally proved in [14]. See also [12, Theorem 2.10] and the references therein.

3.4.3 Proof of Theorem 3.3

Proof of Theorem 3.3 in the case $\alpha \in (0,1)$. Assume that B is diagonalizable and $\widetilde{B} \rhd 0$ (strictly). That is, there exists an invertible matrix U and a diagonal matrix D such that $B = U^{-1}DU$.

By Bernstein's theorem 3.5, if $\alpha \in (0,1)$ there exists a finite measure μ on $\mathbb{R}_+$ such that

$$e^{-x^\alpha} = \int_0^\infty e^{-ux}\, d\mu_\alpha(u)\,.$$

By $\widetilde{B} \rhd 0$, the inequality (3.11) is strict and we can plug $x = D^{1/\alpha}|t|$ into this formula. This follows from the fact that the measure μ_α is finite and that $\exp(-\lambda^{1/\alpha}t)$ is bounded for each eigenvalue λ of D. Hence, we have

$$e^{-D|t|^\alpha} = \int_0^\infty e^{-uD^{1/\alpha}t}\, d\mu_\alpha(u)\,.$$

Conjugating both sides of the equality with U, we obtain

$$e^{-B|t|^\alpha} = \int_0^\infty e^{-uB^{1/\alpha}|t|}\, d\mu_\alpha(u), \tag{3.12}$$

since $B^{1/\alpha} = U^{-1}D^{1/\alpha}U$. We have thus obtained the following representation of f:

$$f(t) = \int_0^\infty \left[e^{-uB^{1/\alpha}|t|}\mathbb{1}_{t\geq 0} + e^{-uB^{1/\alpha,\top}|t|}\mathbb{1}_{t\leq 0}\right] d\mu_\alpha(u)\,.$$

Let us compute Fourier transforms of both sides:

$$\mathcal{F}\left[\exp\left(-uB^{1/\alpha}|t|\right)\,\mathbb{1}_{t\geq 0}\right](\xi) = U^{-1}\mathcal{F}\left[\exp\left(-uD^{1/\alpha}|t|\right)\,\mathbb{1}_{t\geq 0}\right](\xi)\,U$$

and by

$$\mathcal{F}\left[e^{-\lambda t}\,\mathbb{1}_{t\geq 0}\right](\xi) = \frac{1}{\lambda - i\xi}, \quad \text{for} \quad \operatorname{Re}\lambda > 0$$

combined with the fact that $\widetilde{B} \rhd 0$ implies that all eigenvalues λ of B satisfy $|\operatorname{Im}\lambda| < \operatorname{Re}\lambda \cdot |\tan(\pi\alpha/2)|$, we find that

$$\mathcal{F}\left[\exp\left(-uB^{1/\alpha}|t|\right)\,\mathbb{1}_{t\geq 0}\right](\xi) = U^{-1}\left(uD^{1/\alpha} - i\xi\right)^{-1}U = \left(uB^{1/\alpha} - i\xi\right)^{-1}.$$

Hence,

$$\begin{aligned}
\mathcal{F}\left[f(t)\right](\xi) &= \int_0^\infty \left[\left(uB^{1/\alpha} - i\xi\right)^{-1} + \left(uB^{1/\alpha,\top} + i\xi\right)^{-1}\right] d\mu_\alpha(u) \\
&= \int_0^\infty u\left(uB^{1/\alpha} - i\xi\right)^{-1}\left[B^{1/\alpha} + B^{1/\alpha,\top}\right]\left(uB^{1/\alpha} - i\xi\right)^{-1,*} d\mu_\alpha(u)
\end{aligned}$$

By operator-valued Bochner's theorem 3.4, $f(t) \succeq 0$ if and only if its Fourier transform is a positive definite matrix for each ξ. Setting $\xi = 0$ we obtain

$$\mathcal{F}\left[f(t)\right](0) = \left(B^{1/\alpha}\right)^{-1}\left[B^{1/\alpha} + B^{1/\alpha,\top}\right]\left(B^{1/\alpha}\right)^{-1,\top}\int_0^\infty u^{-1}d\mu_\alpha(u).$$

This matrix is positive-definite if and only if

$$B^{1/\alpha} + B^{1/\alpha,\top} \trianglerighteq 0, \tag{3.13}$$

hence this is a necessary condition for the positive definiteness of f. Note however that if this condition is satisfied, then

$$\left(uB^{1/\alpha} - i\xi\right)^{-1} \left[B^{1/\alpha} + B^{1/\alpha,\top}\right] \left(uB^{1/\alpha,\top} - i\xi\right)^{-1,*} \trianglerighteq 0$$

for each ξ and $\mathcal{F}[f](\xi) \trianglerighteq 0$ pointwise. $\square$

Proof of Theorem 3.3 in the case $\alpha \in [1, 2)$. As in the proof for the case $\alpha \in (0, 1)$, our approach is to find an appropriate representation for the Fourier transform of f. Assume that $\xi \geq 0$. Also assume for now that $\widetilde{B} \triangleright 0$. Then there exists $\theta \in (1 - 1/\alpha, 1/\alpha)$ such that $i^{\alpha\theta}B + i^{-\alpha\theta}B^\top \triangleright 0$. The following formula

$$\mathcal{F}\left[e^{-Bt^\alpha}\mathbb{1}_{t\geq 0}\right](\xi) = \int_0^\infty e^{it\xi I - Bt^\alpha}\, dt = i^\theta \int_0^\infty e^{-i^{\theta-1}t\xi I - Bt^\alpha i^{\alpha\theta}}\, dt \tag{3.14}$$

is proven in the Appendix. Performing a change of variables $s = t^\alpha$, we get

$$\mathcal{F}\left[e^{-Bt^\alpha}\mathbb{1}_{t\geq 0}\right](\xi) = i^\theta \int_0^\infty \frac{s^{1/\alpha - 1}}{\alpha}\, e^{-i^{\theta-1}s^{1/\alpha}\xi I - Bi^{\alpha\theta}s}\, ds\,.$$

By Bernstein's Theorem 3.5 there exists a measure μ on $(0, \infty)$ such that:

$$\frac{s^{1/\alpha-1}}{\alpha}\, e^{-s^{1/\alpha}} = \int_0^\infty e^{-us}\, d\mu(u)\,.$$

Plugging $s \rightsquigarrow i^{\alpha(\theta-1)}\xi^\alpha s$, we obtain

$$i^\theta\, \frac{s^{1/\alpha-1}}{\alpha}\, e^{-i^{\theta-1}s^{1/\alpha}\xi} = i^{1-\alpha+\alpha\theta}\xi^{\alpha-1} \int_0^\infty e^{-ui^{\alpha(\theta-1)}\xi^\alpha s}\, d\mu(u)\,.$$

The last step may be justified by using the fact that $\operatorname{Re} i^{\alpha(\theta-1)} > 0$ for $\alpha \in [1, 2)$. Proceeding with the computation above, we find

$$\mathcal{F}\left[e^{-Bt^\alpha}\mathbb{1}_{t\geq 0}\right](\xi) = i^{1-\alpha+\alpha\theta}\xi^{\alpha-1} \int_0^\infty \int_0^\infty e^{-ui^{\alpha(\theta-1)}\xi^\alpha sI - i^{\alpha\theta}Bs}\, ds\, d\mu(u).$$

Now, we can take the integral in s and obtain

$$\mathcal{F}\left[e^{-Bt^\alpha}\mathbb{1}_{t\geq 0}\right](\xi) = \xi^{\alpha-1} \int_0^\infty \left(ui^{-1}\xi^\alpha I + i^{\alpha-1}B\right)^{-1} d\mu(u)$$

for $\xi \geq 0$. Similarly,

$$\mathcal{F}\left[e^{-B^\top(-t)^\alpha}\mathbb{1}_{t<0}\right](\xi) = \xi^{\alpha-1} \int_0^\infty \left(ui^{-1}\xi^\alpha I + i^{\alpha-1}B\right)^{-1,*} d\mu(u)\,.$$

Combining the last two formulas together, we arrive at

$$\mathcal{F}\left[f(t)\right](\xi) = \xi^{\alpha\theta} \int_0^\infty \left(ui^{-1}\xi^\alpha I + i^{\alpha-1}B\right)^{-1} \left[i^{\alpha-1}B + i^{1-\alpha}B^\top\right] \left(ui^{-1}\xi^\alpha I + i^{\alpha-1}B\right)^{-1,*} d\mu(u)\,.$$

Note that

$$\left(ui^{-1}\xi^\alpha I + i^{\alpha-1}B\right)^{-1} \left[i^{\alpha-1}B + i^{1-\alpha}B^\top\right] \left(ui^{-1}\xi^\alpha I + i^{\alpha-1}B\right)^{-1,*} \trianglerighteq 0$$

if and only if

$$i^{\alpha-1}B + i^{1-\alpha}B^\top \trianglerighteq 0,$$

which we have already shown to be true.

If $\widetilde{B} \trianglerighteq 0$, but not necessarily $\widetilde{B} \triangleright 0$, let $B_\varepsilon := B + \varepsilon I$ for $\varepsilon > 0$ and remark that $\widetilde{B_\varepsilon} \triangleright 0$. By the above, we have that

$$f_\varepsilon(t) := \exp\left(-|t|^\alpha \left[B_\varepsilon^+ + B_\varepsilon^- \operatorname{sign}(t)\right]\right)$$

is positive definite for each $\varepsilon > 0$. Letting $\varepsilon \downarrow 0$, we find that f is also positive definite. □

Proof of Theorem 3.3 in the case $\alpha = 2$. In this case (3.4) yields $-iB^- \trianglerighteq 0$. By conjugation we also obtain $iB^- \trianglerighteq 0$. Therefore, $\boldsymbol{z}^*(iB^-)\boldsymbol{z} = 0$ for all $\boldsymbol{z} \in \mathbb{C}^d$. Since iB^- is Hermitian, we can conclude that $B^- = 0$, therefore, $B = B^+$ is Hermitian, in particular, it is normal, and the positive definiteness of f follows from Theorem 3.1. □

3.4.4 Lifting the diagonalizability and strict positive definiteness assumptions

Proof of Theorem 3.3 in the non-diagonalizable case. If $\widetilde{B} \triangleright 0$, but B is not diagonalizable, then there exist diagonalizable matrices B_n converging to B as $n \to \infty$ such that the eigenvalues λ of B_n satisfy the strict inequality

$$|\operatorname{Im}\lambda| < \operatorname{Re}\lambda \cdot \left|\tan\left(\frac{\pi\alpha}{2}\right)\right|. \tag{3.15}$$

Hence,

$$e^{-B_n|t|^\alpha} = \int_0^\infty e^{-uB_n^{1/\alpha}|t|}\, d\mu_\alpha(u),$$

which implies (3.12) by passing to a limit as $n \to \infty$. Having deduced (3.12), we can continue the proof the same way as if B were diagonalizable. □

Proof of Theorem 3.3 in the case when the condition $\widetilde{B} \trianglerighteq 0$ is non-strict. Take $\varepsilon > 0$ and let $B_\varepsilon := (B^{1/\alpha} + \varepsilon I)^\alpha$. The eigenvalues of B_ε satisfy the strict inequality (3.15), and therefore $g_\varepsilon(t) := \exp\left(-|t|^\alpha \left[B_\varepsilon^+ + B_\varepsilon^- \operatorname{sign}(t)\right]\right)$ is positive definite if and only if $B_\varepsilon^{1/\alpha} + B_\varepsilon^{1/\alpha,\top} \trianglerighteq 0$.

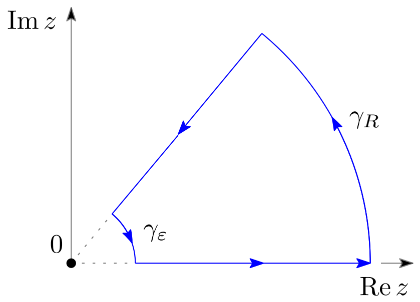
If $B^{1/\alpha} + B^{1/\alpha,\top} \trianglerighteq 0$, then for all $\varepsilon > 0$ $B_\varepsilon^{1/\alpha} + B_\varepsilon^{1/\alpha,\top} \trianglerighteq 0$ and $g_\varepsilon(t) \succeq 0$. Letting $\varepsilon \downarrow 0$, we obtain $f(t) \succeq 0$ as desired.

If $B^{1/\alpha}+B^{1/\alpha,\top}\trianglerighteq 0$ does not hold, then for all sufficiently small $\varepsilon > 0$ $(B+\varepsilon I)^{1/\alpha}+(B+\varepsilon I)^{1/\alpha,\top}\trianglerighteq 0$ also does not hold, but $\widetilde{B}_\varepsilon \triangleright 0$, therefore, f_ε is not positive definite, but then f is not positive definite, because otherwise f_ε would be positive definite as a product of a positive definite matrix-valued function f and a scalar positive definite function $\exp(-\varepsilon|t|^\alpha)$. □

3.5 Appendix

3.5.1 Contour rotation in the proof of case $\alpha \in (1, 2]$

Proof of (3.14). Assume that $\operatorname{Re} i^{\alpha\theta}\lambda > 0$ and $\xi \geq 0$. By Cauchy theorem applied to the following contour γ



Closed contour γ and two its circular arcs γ_ε and γ_R.
The angle at the origin equals $\arg i^\theta = \pi\theta/2$.

we have that for $0 < \varepsilon < R$ holds

$$\oint_\gamma e^{it\xi - \lambda t^\alpha}\, dt = 0,$$

implying

$$\int_\varepsilon^R e^{it\xi-\lambda t^\alpha}\, dt = \int_{i^\theta \varepsilon}^{i^\theta R} e^{it\xi-\lambda t^\alpha}\, dt - \left[\int_{\gamma_\varepsilon} + \int_{\gamma_R}\right] e^{it\xi-\lambda t^\alpha}\, dt\,.$$

Since the integral over γ_ε clearly tends to zero as $\varepsilon \to 0$, and the function under the integral is exponentially small on γ_R, we have

$$\lim_{R\to\infty} \int_{\gamma_R} e^{it\xi-\lambda t^\alpha}\, dt = 0.$$

By changing the variable $t \rightsquigarrow i^\theta t$, we obtain

$$\int_0^\infty e^{it\xi-\lambda t^\alpha}\, dt = \int_{i^\theta \mathbb{R}_+} e^{it\xi-\lambda t^\alpha}\, dt = i^\theta \int_0^\infty e^{-i^{\theta-1}t\xi - i^{\alpha\theta}\lambda t^\alpha}\, dt$$

establishing the proof. □

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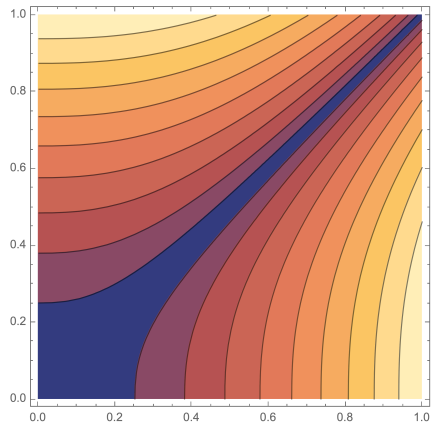
Chapter 4

Extremes of locally-homogenous vector-valued Gaussian processes



Contour plot of a locally homogenous covariance function.

In this chapter, we study the asymptotical behaviour of high exceedence probabilities for centered continuous $\mathbb{R}^n$-valued Gaussian random field $\boldsymbol{X}$ with covariance matrix satisfying

$$\Sigma - R(t+s,t) \sim \sum_{l=1}^{n} B_l(t)\, |s_l|^{\alpha_l} \quad \text{as} \quad s \downarrow 0.$$

Such processes occur naturally as time transformations of homogenous random fields, and we present two asymptotic results of this nature as applications of our findings. The technical novelty of our proof consists in showing that the Slepian-Gordon inequality technique, essential in the univariate case, can also be successfully applied in the multivariate setup. This is noteworthy because this technique was previously believed to be inaccessible in this particular context.

Pavel Ievlev, *Extremes of locally-homogenous vector-valued Gaussian processes*, Extremes **27** (2024), no. 2, 219–245. MR4744268

4.1 Introduction

Despite the fact that the Gaussian extremes have been an active research area since at least the 60s, up until recently little has been known about exact asymptotics of high exceedance probabilities of Gaussian processes *in the multivariate case.* A deep contribution [1] has paved a way towards different problems of the following kind:

$$\mathbb{P}\left\{\exists\, t \in [0,T] \colon \boldsymbol{X}(t) > u\boldsymbol{b}\right\} \quad \text{as} \quad u \to \infty$$

for $\boldsymbol{b} \in \mathbb{R}^d \setminus (-\infty, 0]^d$ and $\boldsymbol{X}$ being a continuous Gaussian process. Here "$>$" denotes the componentwise (Hadamard) comparison. As it turns out, these problems are much more challenging than the univariate ones due to the lack of several techniques which are crucial for the univariate case. The reader can find the detailed account of this shortage in the introduction to the aforementioned paper. Among these lacking techniques, the authors name the Slepian inequality and mention that its extension in the form of Gordon inequality is thought to be inapplicable if the compontents of $\boldsymbol{X}$ are not independent (see [2] for the i.i.d. case).

In this contribution, we aim to achieve two goals. First, we extend [1, Theorem 2.1] on stationary processes to a certain class of homogenous Gaussian random fields defined on $[0,T]^n$, see Theorem 4.1. Second, we apply this result to the study of locally-homogenous Gaussian random fields. The corresponding result is presented in Theorem 4.2. The crucial step of the second part involves constructing two homogenous processes which stochastically dominate $\boldsymbol{X}$ on short intervals from above and from below. This is done by showing that a certain matrix-valued function is positive definite and subsequently applying the Gordon inequality.

As an application of our findings, we present asymptotic formulas for the time-transformed operator fractional Ornstein-Uhlenbeck process $\boldsymbol{Y}$ defined by the covariance matrix function

$$\mathbb{R}^2 \ni (t,s) \mapsto \exp\Big(-\left|\varphi(t) - \varphi(s)\right|^H\Big),$$

with H a symmetric matrix with eigenvalues from $(0,1]$ and φ a strictly monotone continuously differentiable function. By Proposition 4.1,

$$\mathbb{P}\left\{\exists\, \boldsymbol{Y}(t) > u\boldsymbol{b}\right\} \sim c\, u^{1/h}\, \mathbb{P}\left\{\boldsymbol{Y}(0) > u\boldsymbol{b}\right\},$$

where h is the lowest eigenvalue of H and c is given in the form of an integral of Pickands-type constants over $[0,T]$. This result extends [1, Proposition 3.1]. Another application concerns a class of continuous Gaussian processes associated to the following matrix-valued function:

$$\mathbb{R}^2 \ni (t,s) \mapsto \exp\Big(-\left|t-s\right|^\alpha \Big[B^+ + B^- \operatorname{sign}(t-s)\Big]\Big),$$

where $B^\pm = (B \pm B^\top)/2$ are symmetric and antisymmetric parts of a real $d \times d$ matrix B and $\alpha \in (0,2]$. In an upcoming paper [3] we found the necessary and sufficient conditions on the pair (α, B) under which this function is positive definite (see Lemma 4.3) and thus generates a Gaussian process. Here we present an asymptotic result on the time-transformed version of this process, see Proposition 4.2.

The notion of locally stationary process was introduced by Berman in [4] and its extremes were extensively studied afterwards in the papers by Hüsler [5], Piterbarg [6], Chan and Lai [7] and many others. See also [8, 9, 10] for more recent contributions. Its multivariate counterpart, however, has not been considered so far due to the technical issues. The technique of [1] based on the uniform version of local Pickands lemma may in principle be applied to this class of processes, but it would require much stronger assumptions than those we impose in this contribution. Our result, presented in Theorem 4.2, should appear natural (if not obvious) for the specialist, but it still requires a rigorous proof, which involves imposing the right assumptions on the field $\boldsymbol{X}$.

The applicability of Gordon inequality in this context allows to significantly simplify the study of *classical* multivariate Gaussian extremes. In particular, the technical issue of uniformity in the single and double sums may be resolved by passing to a stationary dominating process. Therefore, besides the results here, we establish a simpler methodology compared to [1] for dealing with non-stationary Gaussian random fields.

We want to point out that one possible direction in which our results can be extended is the family of $\alpha(t)$-locally stationary Gaussian random fields, see [11].

Brief organization of the paper. Main results are presented in Section 4.2 with proofs relegated to Section 4.5. The applications are presented in the Section 4.3. Section 4.4 contains auxiliary results and technical lemmas. Appendix contains several known results taken from [1] and reproduced here for reader's convenience in the adapted form.

4.2 Main results

Before proceeding to the theorems, let us introduce some relevant notation.

Vectors. Throughout the paper points of $\mathbb{R}^d$ are written in bold letters (values of multivariate processes), while points of $[0,T]^n \subset \mathbb{R}^n$ (points of their domain) are written in the regular font. This does not lead to any confusion since their meaning can always be understood from the context, but allows to avoid visual clutter. All operations on vectors in both spaces, unless specified otherwise, are performed component-wise. For example, if t and s belong to $\mathbb{R}^n$, then ts denotes the vector $(t_i s_i)_{i=1,\dots,n}$. Similarly for t/s, e^t, $\lfloor t \rfloor$ and so on denoting vectors with components t_i/s_i, e^{t_i} and $\lfloor t_i \rfloor$ correspondingly. We write $t \ge s$ if $t_i \ge s_i$ for all their coordinates. By abuse of notation, we write $1 = (1,\dots,1) \in \mathbb{R}^n$ and $0 = (0,\dots,0) \in \mathbb{R}^n$. If $s > t$, then $[t,s]$ denotes the box $\{u \colon u_i \in [t_i, s_i]\}$.

Matrices. If $A = (A_{ij})_{i,j=1,\dots,d}$ is a $d \times d$ matrix and I, $J \subset \{1,\dots,d\}$ are two index sets, we write A_{IJ} for the submatrix $(A_{ij})_{i\in I,\, j\in J}$. If $I = J$, we occasionally write A_I instead of A_{II}. $\|A\|$ denotes any fixed norm in the space of $d \times d$ matrices. Our formulas do not depend on the choice of the norm. For $\boldsymbol{w} \in \mathbb{R}^d$, $\operatorname{diag}(\boldsymbol{w})$ stands for the diagonal matrix with entries w_1, w_2, …, w_d on the main diagonal. The notation $A \trianglerighteq 0$ means that A is positive definite and $A \triangleright 0$ means that A is strictly positive definite. If A is a real matrix, denote its symmetric and anti-symmetric parts by $A^{\pm} \coloneqq (A \pm A^\top)/2$.

Other notation. We use lower case constants c_1, c_2, … to denote generic constants used in the proofs, whose exact values are not important and can be changed from line to line. The labeling of the constants starts anew in every proof. Similarly, ϵ_1, ϵ_2, … denote error terms, that is, functions of various variables which are small in some specific sense, always described near the point where they are introduced. Their labeling also starts anew in every proof.

The next two subsections present our results on homogenous and locally homogenous fields.

4.2.1 Homogenous case

Let $\boldsymbol{X}(t)$, $t \in [0,T]^n$ be a centered homogenous and continuous Gaussian random field. Denote its covariance and variance matrices by

$$R(t,s) \coloneqq \mathbb{E}\left\{\boldsymbol{X}(t)\,\boldsymbol{X}^\top(s)\right\} \quad \text{and} \quad \Sigma \coloneqq R(0,0).$$

Homogenity means that for each t and s in $[0,T]^n$

$$\mathbb{E}\left\{\boldsymbol{X}(t)\,\boldsymbol{X}^\top(s)\right\} = \mathbb{E}\left\{\boldsymbol{X}(t-s)\,\boldsymbol{X}^\top(0)\right\} = R(t-s,0),$$

therefore we set in the following $R(t) \coloneqq R(t,0)$. It follows that $R(-t) = R^\top(t)$. The matrix $\Sigma - R(t)$ is positive definite, but not necessarily symmetric. Let $\boldsymbol{b} \in \mathbb{R}^d \setminus (-\infty,0]^d$ and denote by $\widetilde{\boldsymbol{b}}$ and I the unique solution of $\Pi_\Sigma(\boldsymbol{b})$ and its I index set, see Lemma 2.5 for details. Set $\boldsymbol{w} \coloneqq \Sigma^{-1}\,\widetilde{\boldsymbol{b}}$.

In this section we impose the following assumptions:

A1 $\Sigma_{II} - R_{II}(t)$ is strictly positive definite for every $t \in (0,T]$

A2 There exist a collection $\mathbb{B} \coloneqq (B_l)_{l=1,\dots,n}$ of real $d\times d$ matrices and a collection of numbers $\boldsymbol{\alpha} \coloneqq (\alpha_l)_{l=1,\dots,n} \in (0,2]^n$ such that

$$\Sigma - R(t) = \sum_{l=1}^n B_l\,|t_l|^{\alpha_l} + o\left(\sum_{l=1}^n |t_l|^{\alpha_l}\right) \quad \text{as} \quad t\downarrow 0, \tag{A2.1}$$

$$\boldsymbol{w}^\top B_l\,\boldsymbol{w} > 0 \quad \text{for all} \quad l = 1,\dots,n. \tag{A2.2}$$

Remark 4.1. *It follows from* (A2.1) *that*

$$\Sigma - R(t) \sim \sum_{l=1}^n \left[B_l\,|t_l|^{\alpha_l}\,\mathbb{1}_{t_l\geq 0} + B_l\,|t_l|^{\alpha_l}\,\mathbb{1}_{t_l<0}\right]$$

as $t\to 0$ and B_l's satisfy

$$\widetilde{B}_l \coloneqq B_l^+\,\cos\left(\frac{\pi\alpha_l}{2}\right) - iB_l^-\,\sin\left(\frac{\pi\alpha_l}{2}\right) \trianglerighteq 0, \quad \textit{where} \quad B^\pm \coloneqq \frac{B\pm B^\top}{2}. \tag{4.1}$$

From this follows that $B_l^+ \trianglerighteq 0$.

Theorem 4.1. *If $\boldsymbol{X}$ is a centered homogenous and continuous Gaussian random field satisfying Assumptions A1 and A2, then*

$$\mathbb{P}\left\{\exists\, t\in[0,T]^n\colon \boldsymbol{X}(t) > u\boldsymbol{b}\right\} \sim T^n\,\mathcal{H}_{\boldsymbol{\alpha},\mathbb{B},\boldsymbol{w}}\prod_{l=1}^n u^{2/\alpha_l}\,\mathbb{P}\left\{\boldsymbol{X}(0) > u\boldsymbol{b}\right\},$$

where the constant $\mathcal{H}_{\boldsymbol{\alpha},\mathbb{B}}$ is given by

$$\mathcal{H}_{\boldsymbol{\alpha},\mathbb{B},\boldsymbol{w}} \coloneqq \lim_{\Lambda\to\infty}\frac{1}{\Lambda^n}\int_{\mathbb{R}^d} e^{\mathbf{1}^\top\boldsymbol{x}}$$
$$\times\,\mathbb{P}\left\{\exists\, t\in[0,\Lambda]^n\colon \sum_{l=1}^n \operatorname{diag}(\boldsymbol{w})\Big[\mathbf{Y}_l(t_l) - S_{\alpha_l,B_l}(t_l)\,\boldsymbol{w}\Big] > \boldsymbol{x}\right\} d\boldsymbol{x} \in (0,\infty). \tag{4.2}$$

Here $\mathbf{Y}_l$ is a continuous Gaussian process associated to the covariance function

$$R_{\alpha_l,B_l}(t_l,s_l) \coloneqq S_{\alpha_l,B_l}(t_l) + S_{\alpha_l,B_l}(-s_l) - S_{\alpha_l,B_l}(t_l - s_l), \qquad S_{\alpha_l,B_l}(t_l) \coloneqq |t_l|^{\alpha_l}\Big[B\mathbb{1}_{t_l\geq 0} + B^\top\mathbb{1}_{t_l<0}\Big].$$

4.2.2 Locally homogenous case

In this section $\boldsymbol{X}(t)$, $t \in [0,T]^n$ is a centered continuous Gaussian random field with covariance matrix

$$R(t,s) := \mathbb{E}\left\{\boldsymbol{X}(t)\,\boldsymbol{X}^\top(s)\right\}$$

and variance matrix Σ satisfying $R(t,t) = R(0,0) =: \Sigma$. We impose the following assumptions:

B1 $\Sigma_{II} - R_{II}(t)$ is strictly positive definite for every $t \in (0,T]$

B2 There exist a collection $\mathbb{B}(t) := (B_l(t))_{l=1,\dots,n}$ of continuous real $d \times d$ matrix-valued functions and a collection of numbers $\boldsymbol{\alpha} := (\alpha_l)_{l=1,\dots,n} \in (0,2]^n$ such that

$$\Sigma - R(t+s,t) = \sum_{l=1}^{n}\left[B_l(t)\,|s_l|^{\alpha_l}\mathbb{1}_{s_l \ge 0} + B_l^\top(t)\,|s_l|^{\alpha_l}\,\mathbb{1}_{s_l<0}\right] + o\left(\sum_{l=1}^{n}|s_l|^{\alpha_l}\right) \quad \text{as} \quad t \to +0, \tag{B2.1}$$

where small-o is uniform in $t \in [0,T]^n$ and

$$\widetilde{B_l}(t) := B_l^+(t)\cos\left(\frac{\pi\alpha_l}{2}\right) - iB_l^-(t)\sin\left(\frac{\pi\alpha_l}{2}\right) \rhd 0 \quad \text{for all} \quad t \in [0,T]^n. \tag{B2.2}$$

Remark 4.2. *From* (B2.2) *follows that* $\boldsymbol{w}^\top\,B_l(t)\,\boldsymbol{w} > 0$ *for all* $t \in [0,T]^n$.

Theorem 4.2. *If* $\boldsymbol{X}$ *is a centered and continuous Gaussian random field satisfying Assumptions B1 and B2, then*

$$\mathbb{P}\left\{\exists\, t \in [0,T]^n \colon \boldsymbol{X}(t) > u\boldsymbol{b}\right\} \sim \int_{[0,T]^n}\mathcal{H}_{\boldsymbol{\alpha},\mathbb{B}(t),\boldsymbol{w}}\,dt\,\prod_{l=1}^{n}u^{2/\alpha_l}\,\mathbb{P}\left\{\boldsymbol{X}(0) > u\boldsymbol{b}\right\},$$

where the constant $\mathcal{H}_{\boldsymbol{\alpha},\mathbb{B}}$ *is given by* (4.2).

4.3 Examples

4.3.1 Time-transformed operator fractional Ornstein-Ulhenbeck process

Let H be a symmetric matrix with all eigenvalues $h_1,\dots,h_d$ belonging to $(0,1]$ and consider a stationary a.s. continuous $\mathbb{R}^d$-valued Gaussian process $\boldsymbol{X}(t)$, $t \ge 0$ with cmf

$$R(t,s) = \exp\left(-\,|t-s|^{2H}\right), \tag{4.3}$$

where $t^H = \exp(H\ln t)$ for $t > 0$. This process is known in the literature as the operator fractional Ornstein-Uhlenbeck process. In this section we consider its time-transformed version. Specifically, let φ be a continuously differentiable strictly monotone function. Define $\boldsymbol{Y}(t) := \boldsymbol{X}(\varphi(t))$. Let us show that this process is locally stationary in the sense defined above. Since H is symmetric, there exists an orthogonal matrix Q such that $H = Q\ \mathrm{diag}(h_1,\dots,h_d)\,Q^\top$. Hence,

$$R(t+s,t) = I - Q\widetilde{I}Q^\top\,|\varphi(t+s)-\varphi(t)|^{2h} + O\left(|\varphi(t+s)-\varphi(t)|^2\right) \quad \text{as} \quad s \to 0,$$

with $h := \min_{i=1,\dots,d} h_i$ and $[\widetilde{I}\,]_{ij} := \mathbb{1}_{i=j\,\text{and}\,h=h_i}$. Since φ is differentiable, we have

$$R(t+s,t) = I - Q\widetilde{I}Q^\top|\varphi'(t)|^{2h}|s|^{2h} + O\left(|s|^{4h}\right) \quad \text{as} \quad s \to 0.$$

Then (B2) holds with $B(t) := Q\widetilde{I}Q^\top\,|\varphi'(t)|^{2h}$ and $\Sigma = I$. Note that $|\varphi'(t)| > 0$ since φ is strictly monotone. By Theorem 4.2 we have the following result:

Proposition 4.1. *Let $\boldsymbol{Y}(t) = \boldsymbol{X}(\varphi(t))$, $t \in [0,T]$, where φ is a continuously differentiable strictly monotone function and $\boldsymbol{X}(t)$, $t \in \mathbb{R}$ is an operator fO-U process associated to the covariance* (4.3) *with a symmetric matrix H whose eigenvalues belong to* $(0,1]$*. Let $\widetilde{b}_j = \max\{b_j, 0\}$ for $j = 1,\ldots,d$. If $\widetilde{\boldsymbol{b}}^\top Q\widetilde{I}Q^\top\widetilde{\boldsymbol{b}} > 0$, then*

$$\mathbb{P}\left\{\exists\, t \in [0,T] \colon \boldsymbol{Y}(t) > u\boldsymbol{b}\right\} \sim u^{1/h} \int_0^T \mathcal{H}_{2h, Q\widetilde{I}Q^\top |\varphi'(t)|^{2h}, \boldsymbol{w}}\, dt\ \ \mathbb{P}\left\{\boldsymbol{X}(\varphi(0)) > u\boldsymbol{b}\right\}.$$

4.3.2 A Gaussian process with α-homogenous log-covariance

In Chapter 3 show the following result:

Theorem 4.3. *Let B be a real $d \times d$ matrix. If a matrix-valued function R defined by*

$$R(t,s) = \exp\left(-|t-s|^\alpha \Big[B^+ + B^- \operatorname{sign}(t-s)\Big]\right), \qquad t,\, s \in \mathbb{R}, \tag{4.4}$$

is positive-definite, then the condition (4.1) *is satisfied. If, on the other hand, the condition* (4.1) *is satisfied. Then*

- *If $\alpha \in (0,1)$, then R is positive-definite if and only if B satisfies*

$$B^{1/\alpha} + B^{1/\alpha,\top} \trianglerighteq 0.$$

- *If $\alpha \in [1,2]$, then R is positive-definite.*

Using the above result, define $\boldsymbol{X}(t)$, $t \in \mathbb{R}$ a stationary continuous Gaussian process associated to this covariance and let φ be a strictly increasing continuously differentiable function. Define $\boldsymbol{Y}(t) \coloneqq \boldsymbol{X}(\varphi(t))$. The covariance of $\boldsymbol{Y}$ satisfies

$$R_{\boldsymbol{Y}}(t+s,t) \sim I - \Big[B^+ + B^- \operatorname{sign}(s)\Big]\, |\varphi'(t)|^\alpha\, |s|^\alpha + O\left(|s|^{2\alpha}\right) \quad \text{as} \quad s \to 0,$$

where we used the fact that $\operatorname{sign}(\varphi(t+s) - \varphi(t)) = \operatorname{sign}(s)$ since φ is increasing. Hence, the assumption B2.1 is satisfied with $B(t) = B\,|\varphi'(t)|^\alpha$. The validity of B2.2 follows from the fact that $|\varphi'(t)| > 0$ and our assumption on B. By Theorem 4.2, we have the following result:

Proposition 4.2. *Let $\boldsymbol{Y}(t) = \boldsymbol{X}(\varphi(t))$, $t \in [0,T]$, where φ is a strictly increasing continuously differentiable function and $\boldsymbol{X}$ is a process associated to the covariance* (4.4)*, where B and α are such that this function is positive definite. Then*

$$\mathbb{P}\left\{\exists\, t \in [0,T] \colon \boldsymbol{Y}(t) > u\boldsymbol{b}\right\} \sim u^{2/\alpha} \int_0^T \mathcal{H}_{\alpha, B|\varphi'(t)|^\alpha, \boldsymbol{w}}\, dt\ \ \mathbb{P}\left\{\boldsymbol{X}(\varphi(0)) > u\boldsymbol{b}\right\}$$

as $u \to \infty$.

4.4 Auxiliary results

4.4.1 Lemma on positive definiteness

Lemma 4.1. *Let B be a real $d \times d$ matrix satisfying*

$$\widetilde{B} = B_+ \sin\left(\frac{\pi\alpha}{2}\right) - iB_- \cos\left(\frac{\pi\alpha}{2}\right) \triangleright 0. \tag{4.5}$$

Then there exists a collection of complex numbers $\{\lambda_k\}_{k=1,\dots,d}$ *satisfying*

$$\operatorname{Re}\lambda_k = 1, \qquad |\operatorname{Im}\lambda_k| < \left|\tan\left(\frac{\pi\alpha}{2}\right)\right| \tag{4.6}$$

and a collection of strictly positive definite Hermitian matrices $\{V_k\}_{k=1,\dots,d}$ *of rank one such that*

$$B = \sum_{k=1}^{d} \lambda_k V_k. \tag{4.7}$$

Proof. Note that B can be represented as follows:

$$B = B_+ + iB'_-, \qquad B'_- := -iB_-, \qquad B_\pm := \frac{B \pm B^\top}{2}.$$

Here B_+ is symmetric and strictly positive definite by (4.5) and B'_- is Hermitian. Hence, there exists an invertible real matrix A such that $B_+ = AA^\top$. Note that for each unitaty matrix Q holds

$$QA^{-1}B_+A^{-\top}Q^* = QQ^* = I.$$

Since B'_- is Hermitian, so is $A^{-1}B'_-A^{-\top}$ and therefore there exists a unitary matrix Q and a real diagonal matrix D such that

$$A^{-1}B'_-A^{-\top} = Q^*DQ.$$

Denote $V := AQ^*$. Therefore, we have the following representations of B_+

$$VV^* = AQ^*QA^\top = AA^\top = B_+ \tag{4.8}$$

and B'_-

$$VDV^* = AQ^*DQA^\top = AA^{-1}B'_-A^{-\top}A^\top = B'_-. \tag{4.9}$$

Hence, for B we have

$$B = B_+ + iB'_- = VV^* + iVDV^* = V\Big[I + iD\Big]V^*.$$

Set next

$$\lambda_k := 1 + iD_{kk}, \qquad V_k := V\,\mathcal{D}_k\,V^*, \tag{4.10}$$

where $[\mathcal{D}_k]_{ml} = \delta_{km}\delta_{kl}$ is the diagonal matrix with 1 at k-th place. Clearly, V_k's are Hermitian, positive definite, of rank one and (4.7) is satisfied. It remains to show that the inequality (4.6) is also satisfied. To this end, use (4.8) and (4.9) to rewrite $\widetilde{B}$ as

$$\widetilde{B} = V\left[I\cos\left(\frac{\pi\alpha}{2}\right) - iD\sin\left(\frac{\pi\alpha}{2}\right)\right]V^* \rhd 0.$$

Therefore, we have

$$I\cos\left(\frac{\pi\alpha}{2}\right) - iD\sin\left(\frac{\pi\alpha}{2}\right) \rhd 0,$$

which implies (4.6). □

Lemma 4.2. *Under the conditions of Lemma 4.1, the functions given by*

$$\mathcal{E}_{\alpha,B,k}(t) := \exp\left(-d\lambda_k V_k|t|^\alpha\right)\mathbb{1}_{t\geq 0} + \exp\left(-d\,\overline{\lambda}_k\,V_k|t|^\alpha\right)\mathbb{1}_{t<0}$$

with λ_k, V_k and α from Lemma 4.1 are all positive definite complex matrix-valued functions. Let $\Sigma = AA^\top$ be a strictly positive definite matrix and define

$$\mathcal{E}_{\alpha,B}(t) \coloneqq \frac{1}{2d} A \sum_{k=1}^{d} \Big[\mathcal{E}_{\alpha,A^{-1}BA^{-\top},k}(t) + \overline{\mathcal{E}_{\alpha,A^{-1}BA^{-\top},k}(t)}\Big] A^\top.$$

Then $\mathcal{E}_{\alpha,B}(t)$ is positive definite real matrix-valued function satisfying

$$\mathcal{E}_{\alpha,B}(t) = \Sigma - B|t|^\alpha \mathbb{1}_{t\ge 0} - B^\top |t|^\alpha \mathbb{1}_{t<0} + o\left(|t|^\alpha\right) \quad \text{as} \quad t \to 0.$$

Proof. Since $V_k = V^* \mathcal{D}_k V$ by (4.10), there exists $\mu_k > 0$ and a unitary matrix U such that $V_k = \mu_k U^* \mathcal{D}_k U$. Hence,

$$\exp\left(-d\big[1 + i \operatorname{Im}\lambda_k \operatorname{sign}(t)\big] V_k |t|^\alpha\right) = U^* \exp\left(-d\mu_k \Big[1 + i \operatorname{Im}\lambda_k \operatorname{sign}(t)]\mathcal{D}_k\, |t|^\alpha\right) U.$$

Positive definiteness of this function is therefore equivalent to that of a scalar-valued function

$$\exp\left(-d\mu_k \Big[1 + i \operatorname{Im}\lambda_k \ \operatorname{sign}(t)\Big] |t|^\alpha\right),$$

which follows from (4.6). The second claim follows from (4.7) and the fact that

$$\widetilde{B} \rhd 0 \implies \widetilde{A^{-1}BA^{-\top}} = A^{-1}\widetilde{B}A^{-\top} \rhd 0$$

by a direct computation. □

4.4.2 Double sum bound

Lemma 4.3 (Double sum bound)**.** *If $\boldsymbol{X}(t)$, $t \in [0,T]^n$ is a centered continuous Gaussian field satifying Assumption A2, then there exist positive constants C and ε such that for every $k \in \mathbb{Z}^d \setminus \{0\}$ with $1 < |k_l| \le N_u(\varepsilon)$ for all l and $\Lambda > 0$ holds*

$$\frac{P_{\boldsymbol{b}}(k,\Lambda)}{\mathbb{P}\left\{\boldsymbol{X}(0) > u\boldsymbol{b}\right\}} \le C \Lambda^{\#\{l\colon k_l = 0\}} \prod_{l\colon k_l \ne 0} (|k_l| - 1)^{-2} \exp\left(-\frac{1}{4} \boldsymbol{w}^\top B_l\, \boldsymbol{w}\, \Lambda^{\alpha_l} \left(|k_l| - 1\right)^{\alpha_l}\right)$$

Remark 4.3. *Note that the conditions of the lemma demand that there be no l's such that $k_l = \pm 1$. This is not a coincidence: the adjacent double events are to be estimated differently. See the proof of Theorem 4.1 for details.*

Proof. Without loss of generality assume that $I = \{1, \ldots, n\}$. Then

$$\begin{aligned} P_{\boldsymbol{b}}(k,\Lambda) &\le \mathbb{P}\left\{\exists\, (t,s) \in \Lambda u^{-2/\alpha}[k, k+1] \times [0,1] \colon \frac{1}{2}\Big[\boldsymbol{X}(t) + \boldsymbol{X}(s)\Big] > u\boldsymbol{b}\right\} \\ &= u^{-d} \int_{\mathbb{R}^d} \mathbb{P}\left\{\exists\, (t,s) \in [0,\Lambda]^{2n} \colon \boldsymbol{\chi}_{u,k,\boldsymbol{x}}(t,s) > \boldsymbol{x}\right\} \varphi_{u,k}\left(u\boldsymbol{b} - \frac{\boldsymbol{x}}{u}\right) d\boldsymbol{x}, \end{aligned} \tag{4.11}$$

where

$$\boldsymbol{\chi}_{u,k,\boldsymbol{x}}(t,s) \coloneqq u\left(\boldsymbol{X}_{u,k}(t,s) - u\boldsymbol{b} \,\middle|\, \boldsymbol{X}_{u,k}(0,0) = u\boldsymbol{b} - \frac{\boldsymbol{x}}{u}\right) + \boldsymbol{x}$$

with

$$\boldsymbol{X}_{u,k}(t,s) \coloneqq \frac{1}{2}\Big[\boldsymbol{X}\left(\Lambda u^{-2/\alpha} k + u^{-2/\alpha} t\right) + \boldsymbol{X}\left(u^{-2/\alpha} s\right)\Big]$$

and $\varphi_{u,k}$ is the pdf of $\boldsymbol{X}_{u,k}(0,0) \overset{d}{=} N(0, \Sigma_{u,k})$, where

$$\begin{aligned}\Sigma_{u,k} &:= \mathbb{E}\left\{\boldsymbol{X}_{u,k}(0,0)\, \boldsymbol{X}_{u,k}^\top(0,0)\right\} = \frac{1}{4}\Big[2\Sigma + R\left(\Lambda u^{-2/\alpha} k\right) + R\left(-\Lambda u^{-2/\alpha} k\right)\Big] \\ &= \Sigma - u^{-2} \sum_{l=1}^{n} \left[B_l + B_l^\top\right] \Lambda^{\alpha_l} k_l^{\alpha_l} + \epsilon\left(u^{-2/\alpha} \Lambda k\right).\end{aligned} \tag{4.12}$$

First, bound $\varphi_{u,k}$ as follows:

$$\varphi_{u,k}\left(u\boldsymbol{b} - \frac{\boldsymbol{x}}{u}\right) \le \varphi(u\boldsymbol{b}) \exp\left(\frac{u^2}{2}\, \boldsymbol{b}^\top \left[\Sigma^{-1} - \Sigma_{u,k}^{-1}\right] \boldsymbol{b}\right) \exp\left(\boldsymbol{b}^\top \Sigma_{u,k}^{-1} \boldsymbol{x}\right),$$

where φ is the pdf of $N(0, \Sigma)$. Plugging this into (4.11) and noting that $u^{-d}\, \varphi(u\boldsymbol{b}) = \mathbb{P}\left\{\boldsymbol{X}(0) > u\boldsymbol{b}\right\}$, we obtain the following bound:

$$\begin{aligned}\frac{P_{\boldsymbol{b}}(k, \Lambda)}{\mathbb{P}\left\{\boldsymbol{X}(0) > u\boldsymbol{b}\right\}} &\le \exp\left(\frac{u^2}{2}\, \boldsymbol{b}^\top \left[\Sigma^{-1} - \Sigma_{u,k}^{-1}\right] \boldsymbol{b}\right) \\ &\qquad \times \int_{\mathbb{R}^d} \exp\left(\boldsymbol{b}^\top \Sigma_{u,k}^{-1} \boldsymbol{x}\right) \mathbb{P}\left\{\exists\, (t,s) \in [0, \Lambda]^{2n} \colon \boldsymbol{\chi}_{u,k,\boldsymbol{x}}(t,s) > \boldsymbol{x}\right\} d\boldsymbol{x}.\end{aligned} \tag{4.13}$$

At this point we split the proof into three parts: estimation of the integral, estimation of the exponent in front of it and their comparison.

The exponent in front of the integral. By (4.12), we have

$$\Sigma^{-1} - \Sigma_{u,k}^{-1} = -u^{-2} \sum_{l=1}^{n} \Sigma^{-1} \left[B_l + B_l^\top\right] \Sigma^{-1} \Lambda^{\alpha_l} |k_l|^{\alpha_l} + \epsilon_1\left(u^{-2/\alpha} \Lambda k\right). \tag{4.14}$$

Therefore,

$$\frac{u^2}{2}\, \boldsymbol{b}^\top \left[\Sigma^{-1} - \Sigma_{u,k}^{-1}\right] \boldsymbol{b} = -\sum_{l=1}^{n} \boldsymbol{w}^\top B_l\, \boldsymbol{w}\, \Lambda^{\alpha_l} |k_l|^{\alpha_l} + u^2 \epsilon_2\left(u^{-2/\alpha} \Lambda k\right). \tag{4.15}$$

By our assumptions,

$$\sup_{-N_u(\varepsilon) \le k \le N_u(\varepsilon)} u^2 \left|\epsilon_2\left(u^{-2/\alpha} \Lambda k\right)\right| \xrightarrow[u \to \infty]{} 0.$$

The integral. First note that

$$\exp\left(\boldsymbol{b}^\top \Sigma_{u,k}^{-1} \boldsymbol{x}\right) = \exp\left((\boldsymbol{w} + \epsilon_3(u^{-2/\alpha} \Lambda k))^\top \boldsymbol{x}\right)$$

where the ϵ_3 term tends to zero uniformly in k. We will drop this term from now on to simplify the notation. To bound the remaining integral we will use Lemma 2.10, which gives

$$\int_{\mathbb{R}^d} e^{\boldsymbol{w}^\top \boldsymbol{x}}\, \mathbb{P}\left\{\exists\, (t,s) \in [0, \Lambda]^{2n} \colon \boldsymbol{\chi}_{u,k,\boldsymbol{x}}(t,s) > u\boldsymbol{b}\right\} d\boldsymbol{x} \le c_1 \exp\left(c_2(G + \sigma^2)\right) \tag{4.16}$$

with some positive constants c_1 and c_2. Here $G \in \mathbb{R}$ and $\sigma^2 > 0$ are numbers (depending on k and u) such that

$$\sup_{F \subset \{1, \ldots, d\}}\ \sup_{(t,s) \in [0, \Lambda]^{2n}} \boldsymbol{w}_F^\top\, \mathbb{E}\left\{\boldsymbol{\chi}_{u,k,\boldsymbol{x},F}(t,s)\right\} \le G + \varepsilon \sum_{j=1}^{d} |x_j| \tag{4.17}$$

and

$$\sup_{F\subset\{1,\dots,d\}}\ \sup_{(t,s)\in[0,\Lambda]^{2n}} \mathrm{Var}\left\{\boldsymbol{w}_F^\top \boldsymbol{\chi}_{u,k,\boldsymbol{x},F}(t,s)\right\} \le \sigma^2$$

To apply this lemma we need to find such numbers.

Finding G. By the formulas on conditional Gaussian distribution, we have

$$\mathbb{E}\left\{\boldsymbol{\chi}_{u,k,\boldsymbol{x}}(t,s)\right\} = u\Big[\Sigma_{u,k} - R_{u,k}(t,s,0,0)\Big]\Sigma_{u,k}^{-1}\left[u\boldsymbol{b} - \frac{\boldsymbol{x}}{u}\right], \tag{4.18}$$

where $R_{u,k}(t,s,t',s')$ is the covariance of $\boldsymbol{\chi}_{u,k,\boldsymbol{x}}(t,s)$. Note that this covariance does not depend on $\boldsymbol{x}$. The $\boldsymbol{x}$-term can clearly be bounded by

$$\left\|\Big[\Sigma_{u,k} - R_{u,k}(t,s,0,0)\Big]\Sigma_{u,k}^{-1}\boldsymbol{x}\right\| \le \varepsilon\sum_{j=1}^{d}|x_j|^d.$$

Let us bound the $\boldsymbol{b}$-contribution. A direct computation gives

$$\begin{aligned}\Sigma_{u,k} - R_{u,k}(t,s,0,0) \sim \frac{1}{4u^2}\sum_{l=1}^{n}\Big[&S_{\alpha_l,B_l}(s_l) + S_{\alpha_l,B_l}(t_l)\\ &+ S_{\alpha_l,B_l}(\Lambda k_l + t_l) + S_{\alpha_l,B_l}(s_l - \Lambda k_l) - S_{\alpha_l,B_l}(-\Lambda k_l) - S_{\alpha_l,B_l}(\Lambda k_l)\Big]\end{aligned} \tag{4.19}$$

uniformly in $k\in[-N_u(\varepsilon),N_u(\varepsilon)]$. By (4.14)

$$\begin{aligned}u^2\,\boldsymbol{w}_F^\top\left[\Big[\Sigma_{u,k} - R_{u,k}(t,s,0,0)\Big]\Sigma_{u,k}^{-1}\boldsymbol{b}\right]_F &\sim u^2\,\boldsymbol{w}_F^\top\left[\Big[\Sigma_{u,k} - R_{u,k}(t,s,0,0)\Big]\boldsymbol{w}\right]_F\\ &\sim \frac{1}{4}\sum_{l=1}^{n}[A_{1,l} + A_{2,l} + A_{3,l}]\end{aligned}$$

uniformly in $k\in[-N_u(\varepsilon),N_u(\varepsilon)]$, where

$$\begin{aligned}A_{1,l} &:= \boldsymbol{w}_F^\top\left[\Big[S_{\alpha_l,B_l}(s_l) + S_{\alpha_l,B_l}(t_l)\Big]\boldsymbol{w}\right]_F,\\ A_{2,l} &:= \boldsymbol{w}_F^\top\left[\Big[S_{\alpha_l,B_l}(\Lambda k_l + t_l) - S_{\alpha_l,B_l}(\Lambda k_l)\Big]\boldsymbol{w}\right]_F,\\ A_{3,l} &:= \boldsymbol{w}_F^\top\left[\Big[S_{\alpha_l,B_l}(s_l - \Lambda k_l) - S_{\alpha_l,B_l}(-\Lambda k_l)\Big]\boldsymbol{w}\right]_F.\end{aligned}$$

The first can be bounded as follows:

$$|A_{1,l}| \le |\boldsymbol{w}|^2\Big[\,\|S_{\alpha_l,B_l}(s_l)\| + \|S_{\alpha_l,B_l}(t_l)\|\,\Big] \le 2\Lambda^{\alpha_l}\,|\boldsymbol{w}|^2\,\|B_l\|\,.$$

$A_{2,l}$ and $A_{3,l}$ can be bounded for $k_l\neq 0$ similarly as follows:

$$|A_{2,l}| \le |\boldsymbol{w}|^2\,\|B\|\,\Big[\,|\Lambda k_l + t_l|^{\alpha_l} - |\Lambda k_l|^{\alpha_l}\,\Big] \le c_2\,\Lambda^{\alpha_l}|k_l|^{\alpha_l-1}.$$

Therefore, the inequality (4.17) is satisfied with

$$G = c_2\sum_{l=1}^{n}\Lambda^{\alpha_l}(1 + |k_l|^{\alpha_l-1}\mathbb{1}_{k_l\neq 0}). \tag{4.20}$$

Finding σ^2. We have

$$\begin{aligned}\operatorname{Var}\left\{\boldsymbol{w}_F^\top \boldsymbol{\chi}_{u,k,\boldsymbol{x},F}(t,s)\right\} &= \sum_{j',\, j\in F} w_j w_{j'} \operatorname{Cov}(\chi_{u,k,\boldsymbol{x},j}(t,s), \chi_{u,k,\boldsymbol{x},j'}(t,s)) \\ &\le c_3 \sum_{j,j'} \Big[\mathcal{R}_{u,k,\boldsymbol{x}}(t,s,t,s)\Big]_{j,j'},\end{aligned}$$

where $\mathcal{R}$ is the covariance of $\boldsymbol{\chi}_{u,k,\boldsymbol{x},F}$:

$$\begin{aligned}\mathcal{R}_{u,k,\boldsymbol{x}}(t,s,t',s') &\coloneqq \mathbb{E}\left\{\boldsymbol{\chi}_{u,k,\boldsymbol{x}}(t,s)\,\boldsymbol{\chi}_{u,k,\boldsymbol{x}}^\top(t',s')\right\} \\ &= R_{u,k}(t,s,t',s') - R_{u,k}(t,s,0,0)\Sigma_{u,k}^{-1} R_{u,k}(0,0,t',s') \\ &\sim \frac{1}{4}\sum_{l=1}^n \Big[A_{1,l} + A_{2,l} + A_{3,l} + A_{4,l} + A_{5,l} + A_{6,l}\Big],\end{aligned}$$

where

$$\begin{aligned}A_{1,l} &\coloneqq S_{\alpha_l,B_l}(t_l) + S_{\alpha_l,B_l}(s_l) + S_{\alpha_l,B_l}(-t_l') + S_{\alpha_l,B_l}(-s_l'), \\ A_{2,l} &\coloneqq S_{\alpha_l,B_l}(s_l - \Lambda k_l) - S_{\alpha_l,B_l}(-\Lambda k_l), \\ A_{3,l} &\coloneqq S_{\alpha_l,B_l}(t_l + \Lambda k_l) - S_{\alpha_l,B_l}(\Lambda k_l), \\ A_{4,l} &\coloneqq -S_{\alpha_l,B_l}(s - s') - S_{\alpha_l,B_l}(t - t'), \\ A_{5,l} &\coloneqq S_{\alpha_l,B_l}(-\Lambda k_l - t_l') - S_{\alpha_l,B_l}(-\Lambda k_l - t_l' + s_l), \\ A_{6,l} &\coloneqq S_{\alpha_l,B_l}(\Lambda k_l - s_l') - S_{\alpha_l,B_l}(\Lambda k_l - s_l' + t_l).\end{aligned}$$

Similarly to how we bounded differences of this form above, we obtain

$$\|A_{1,l}\|\,,\,\|A_{4,l}\| \le c_4 \Lambda^{\alpha_l}, \qquad \|A_{2,l}\|\,,\,\|A_{3,l}\|\,,\,\|A_{5,l}\|\,,\,\|A_{6,l}\| \le c_5 \Lambda^{\alpha_l} |k_l|^{\alpha_l - 1}.$$

Hence, the inequality (4.17) is satisfied with

$$\sigma^2 = c_6 \sum_{l=1}^n \Lambda^{\alpha_l}\left(1 + |k_l|^{\alpha_l-1}\mathbb{1}_{k_l\neq 0}\right). \tag{4.21}$$

Proceeding with the integral. Combining (4.20) and (4.21) with (4.16), we find

$$\int_{\mathbb{R}^d} e^{\boldsymbol{w}^\top \boldsymbol{x}}\, \mathbb{P}\left\{\exists\,(t,s)\in[0,\Lambda]^{2n}\colon \boldsymbol{\chi}_{u,k,\boldsymbol{x}}(t,s) > u\boldsymbol{b}\right\} d\boldsymbol{x} \le c_6 \exp\left(c_7 \sum_{l=1}^n \Lambda^{\alpha_l}\left(1 + |k_l|^{\alpha_l-1}\mathbb{1}_{k_l\neq 0}\right)\right).$$

By (4.15) and (4.13), we have

$$\frac{P_{\boldsymbol{b}}(k,\Lambda)}{\mathbb{P}\left\{\boldsymbol{X}(0) > u\boldsymbol{b}\right\}} \le c_8 \exp\left(-\sum_{l=1}^n \Lambda^{\alpha_l}\left[\frac{\boldsymbol{w}^\top B_l\,\boldsymbol{w}}{2}\,|k_l|^{\alpha_l} - c_7\left(1 + |k_l|^{\alpha_l-1}\mathbb{1}_{k_l\neq 0}\right)\right]\right). \tag{4.22}$$

If $|k_l|$ is large enough, we have

$$\frac{\boldsymbol{w}^\top B_l\,\boldsymbol{w}}{2}\,|k_l|^{\alpha_l} - c_7\left(1 + |k_l|^{\alpha_l}\right) \ge \frac{\boldsymbol{w}^\top B_l\,\boldsymbol{w}}{4}.$$

Lifting the assumption that $|k_l|$ is large. Let K be such that for $|k_l| \geq K$ holds

$$\frac{P_{\boldsymbol{b}}(k,\Lambda)}{\mathbb{P}\left\{\boldsymbol{X}(0) > u\boldsymbol{b}\right\}} \leq c_8 \exp\left(-\frac{1}{4}\sum_{l=1}^{n} \boldsymbol{w}^\top B_l\, \boldsymbol{w}\, \Lambda^{\alpha_l} |k_l|^{\alpha_l}\right).$$

It suffices to consider the case when some of k_l's satisfy $1 < |k_l| < K$. Assume for simplicity that there is exactly one such l that $|k_l| < K$, take $\Lambda' > 0$ such that $\Lambda' < \Lambda$ and bound $P_{\boldsymbol{b}}$ as follows:

$$\begin{aligned} P_{\boldsymbol{b}}(k,\Lambda) &\leq \sum_{0 \leq p_l,\, q_l \leq \lceil \Lambda/\Lambda' \rceil} \mathbb{P}\begin{Bmatrix} \exists\, t \in \Lambda' u^{-2/\alpha}[\Lambda k/\Lambda' + q_l 1_l, \Lambda k/\Lambda' + q_l 1_l + 1]: & \boldsymbol{X}(t) > u\boldsymbol{b} \\ \exists\, s \in \Lambda' u^{-2/\alpha}[p_l 1_l, p_l 1_l + 1]: & \boldsymbol{X}(s) > u\boldsymbol{b} \end{Bmatrix} \\ &= \sum_{0 \leq p_l,\, q_l \leq \lceil \Lambda/\Lambda' \rceil} P_{\boldsymbol{b}}(\Lambda k/\Lambda' + (q_l - p_l) 1_l, \Lambda'). \end{aligned} \tag{4.23}$$

Here $1_l \in \mathbb{Z}^d$ such that $[1_l]_{l'} = \delta_{l,l'}$. Choose $\Lambda' := \Lambda(|k_l| - 1)/K$. Then

$$k_l' := \Lambda k_l/\Lambda' + q_l - p_l \geq \Lambda k_l/\Lambda' - \Lambda/\Lambda' = \Lambda(k_l - 1)/\Lambda' \geq K$$

and therefore

$$\begin{aligned} \frac{P_{\boldsymbol{b}}(k',\Lambda')}{\mathbb{P}\left\{\boldsymbol{X}(0) > u\boldsymbol{b}\right\}} &\leq c_8 \exp\left(-\frac{1}{4}\sum_{l=1}^{n} \boldsymbol{w}^\top B_l\, \boldsymbol{w}\, \Lambda'^{\alpha_l}\, |k_l'|^{\alpha_l}\right) \\ &= c_8 \exp\left(-\frac{1}{4}\sum_{l=1}^{n} \boldsymbol{w}^\top B_l\, \boldsymbol{w}\, \Lambda^{\alpha_l} (|k_l| - 1)^{\alpha_l}\right). \end{aligned} \tag{4.24}$$

It remains to note that the number of terms in the sum (4.23) is at most

$$\lceil \Lambda/\Lambda' \rceil^2 \leq 2K^2/(|k_l| - 1)^2.$$

Lifting the assumption that all k_l's are non-zero. By (4.22) and (4.24)

$$\frac{P_{\boldsymbol{b}}(k,\Lambda)}{\mathbb{P}\left\{\boldsymbol{X}(0) > u\boldsymbol{b}\right\}} \leq c_8 \prod_{l:\, k_l \neq 0} \exp\left(-\frac{1}{4}\boldsymbol{w}^\top B_l\, \boldsymbol{w}\, \Lambda^{\alpha_l}\ (|k_l| - 1)^{\alpha_l}\right) \prod_{l:\, k_l = 0} \exp\left(c_7 \Lambda^{\alpha_l}\right). \tag{4.25}$$

Similarly to the previous point of the proof, take $\Lambda' \in (0,\Lambda)$ and assume for simplicity that there is only one l such that $k_l = 0$. Note that

$$P_{\boldsymbol{b}}(k,\Lambda) \leq \sum_{0 \leq p \leq \lceil \Lambda/\Lambda' \rceil} \mathbb{P}\left\{\begin{matrix} \exists\, t_j \in \Lambda u^{-2/\alpha_j}[k_j, k_j + 1],\ j \neq l \\ \exists\, t_l \in \Lambda' u^{-2/\alpha_l}[p, p+1] \\ \exists\, s_j \in \Lambda u^{-2/\alpha_j}[0, 0+1],\ j \neq l \\ \exists\, s_l \in \Lambda' u^{-2/\alpha_l}[p, p+1] \end{matrix} \begin{matrix} : \quad \boldsymbol{X}(t) > u\boldsymbol{b} \\ \\ : \quad \boldsymbol{X}(s) > u\boldsymbol{b} \end{matrix}\right\} \tag{4.26}$$

A similar proof to what we used above shows that each term of this sum is at most

$$c_8 \prod_{l' \neq l} \exp\left(-\frac{1}{4}\boldsymbol{w}^\top B_l\, \boldsymbol{w}\, \Lambda^{\alpha_l}\ (|k_l| - 1)^{\alpha_l}\right) \exp\left(c_7 \Lambda'^{\alpha_l}\right) \mathbb{P}\left\{\boldsymbol{X}(0) > u\boldsymbol{b}\right\}.$$

The number of terms in the sum (4.26) is at most $\lceil \Lambda/\Lambda' \rceil$, hence

$$\frac{P_{\boldsymbol{b}}(k,\Lambda)}{\mathbb{P}\left\{\boldsymbol{X}(0) > u\boldsymbol{b}\right\}} \leq c_9 \Lambda \prod_{l \neq l'} \exp\left(-\frac{1}{4}\boldsymbol{w}^\top B_l\, \boldsymbol{w}\, \Lambda^{\alpha_l}\ (|k_l|^{\alpha_l} - 1)^{\alpha_l}\right),$$

where $c_9 = 2c_8 \exp(c_7 \Lambda'^{\alpha'})/\Lambda'$. The general case when there is several l's such that $k_l = 0$ can be addressed similarly. ☐

4.5 Proofs

4.5.1 Homogenous theorem

Proof of Theorem 4.1. We begin the proof by splitting $[0,T]^n$ into pieces of Pickands scale

$$[0,T]^n = \Lambda u^{-2/\alpha} \bigcup_{k \le N_u} [k, k+1], \quad \text{where} \quad N_u(T) := \left\lceil \frac{T}{\Lambda u^{-2/\alpha}} \right\rceil$$

and using Bonferroni inequality to obtain

$$\Sigma_1' - \Sigma_2 \le \mathbb{P}\left\{\exists\, t \in [0,T]^n \colon \boldsymbol{X}(t) > u\boldsymbol{b}\right\} \le \Sigma_1,$$

where

$$\Sigma_1 := \sum_{0 \le k \le N_u(T)} \mathbb{P}\left\{\exists\, t \in \Lambda u^{-2/\alpha}[k, k+1] \colon \boldsymbol{X}(t) > u\boldsymbol{b}\right\},$$

$$\Sigma_2 := \sum_{\substack{0 \le k, j \le N_u(T) \\ k \ne j}} \mathbb{P}\left\{\begin{matrix} \exists\, t \in \Lambda u^{-2/\alpha}[k, k+1] \colon \boldsymbol{X}(t) > u\boldsymbol{b} \\ \exists\, s \in \Lambda u^{-2/\alpha}[j, j+1] \colon \boldsymbol{X}(s) > u\boldsymbol{b} \end{matrix}\right\}.$$

and Σ_1' is defined by the same formula as Σ_1 but with $N-1$ instead of N in the upper summation limit. At this point we split the proof into two parts. First, we will focus on finding the exact asymptotics of the single sum $\Sigma_1 \sim \Sigma_1'$, and then demonstrate that the double sum Σ_2 is negligible with respect to Σ_1.

Since $\boldsymbol{X}$ is homogenous, we can easily compute the single sum

$$\Sigma_1 = \left[\prod_{l=1}^{n} N_{u,l}(T)\right] \mathbb{P}\left\{\exists\, t \in \Lambda u^{-2/\alpha}[0,1]^n \colon \boldsymbol{X}(t) > u\boldsymbol{b}\right\}.$$

Applying local Pickands Lemmma 4.5, we obtain

$$\Sigma_1' \sim \Sigma_1 \sim T^n \left[\prod_{l=1}^{n} u^{-2/\alpha_l}\right] \frac{\mathcal{H}_{\boldsymbol{\alpha}, \mathbb{B}, \boldsymbol{w}}([0,\Lambda]^n)}{\Lambda^n} \mathbb{P}\left\{\boldsymbol{X}(0) > u\boldsymbol{b}\right\}.$$

Since $E \mapsto \mathcal{H}_{\boldsymbol{\alpha}, \mathbb{B}, \boldsymbol{w}}(E)$ is subadditive, we have that the limit

$$\mathcal{H}_{\boldsymbol{\alpha}, \mathbb{B}, \boldsymbol{w}} := \lim_{\Lambda \to \infty} \frac{\mathcal{H}_{\boldsymbol{\alpha}, \mathbb{B}, \boldsymbol{w}}([0,\Lambda]^n)}{\Lambda^n}$$

exists and is finite. We will show that it is also positive after dealing with the double sum.

Double sum. By stationarity we have that

$$\Sigma_2 = \sum_{\substack{0 \le k, j \le N_u(T) \\ k \ne j}} \mathbb{P}\left\{\begin{matrix} \exists\, t \in \Lambda u^{-2/\alpha}[k-j, k-j+1] \colon & \boldsymbol{X}(t) > u\boldsymbol{b} \\ \exists\, s \in \Lambda u^{-2/\alpha}[0, 1] \colon & \boldsymbol{X}(s) > u\boldsymbol{b} \end{matrix}\right\}.$$

Reindexing the sum by $q = k - j$, we obtain

$$\Sigma_2 = \prod_{l=1}^{n} N_{u,l}(T) \sum_{\substack{-N_u(T) \le q \le N_u(T) \\ q \ne 0}} \mathbb{P}\left\{\begin{matrix} \exists\, t \in \Lambda u^{-2/\alpha}[q, q+1] \colon & \boldsymbol{X}(t) > u\boldsymbol{b} \\ \exists\, s \in \Lambda u^{-2/\alpha}[0, 1] \colon & \boldsymbol{X}(s) > u\boldsymbol{b} \end{matrix}\right\}.$$

Denote the double events' probabilities by

$$P_{\boldsymbol{b}}(q,\Lambda) \coloneqq \mathbb{P}\left\{\begin{matrix} \exists\, t \in \Lambda u^{-2/\alpha}[q,q+1]\colon & \boldsymbol{X}(t) > u\boldsymbol{b} \\ \exists\, s \in \Lambda u^{-2/\alpha}[0,1]\colon & \boldsymbol{X}(s) > u\boldsymbol{b} \end{matrix}\right\}.$$

Take some $\varepsilon \in (0,T)$ and divide the sum in two parts:

$$\sum_{\substack{0 \le q \le N_u \\ q \ne 0}} P_{\boldsymbol{b}}(q,\Lambda) = \sum_{\exists\, l\colon |q_l| > N_{u,l}(\varepsilon)} P_{\boldsymbol{b}}(q,\Lambda) + \sum_{-N_u(\varepsilon) \le q \le N_u(\varepsilon)} P_{\boldsymbol{b}}(q,\Lambda). \tag{4.27}$$

Terms of the first sum can be bounded as follows:

$$\begin{aligned} P_{\boldsymbol{b}}(q,\Lambda) &\le \mathbb{P}\left\{\exists\,(t,s) \in \Lambda u^{-2/\alpha}([q,q+1]\times[0,1])\colon \frac{1}{2}\left[\boldsymbol{X}(t) + \boldsymbol{X}(s)\right] > u\boldsymbol{b}\right\} \\ &\le \mathbb{P}\left\{\exists\,(t,s) \in \Lambda u^{-2/\alpha}([q,q+1]\times[0,1])\colon \frac{1}{2}\left[\boldsymbol{X}_I(t) + \boldsymbol{X}_I(s)\right] > u\boldsymbol{b}_I\right\}. \end{aligned}$$

Let $\Sigma(t,s)$ denote the variance matrix of $(\boldsymbol{X}(t)+\boldsymbol{X}(s))/2$:

$$\Sigma(t,s) = \frac{1}{4}\Big[2\Sigma + R(t-s) + R(s-t)\Big].$$

In view of Assumption A1, the matrix $(\Sigma_{II}(t,s))^{-1}) - (\Sigma_{II})^{-1}$ is strictly positive definite for $t \ne s$, which implies

$$\begin{aligned} \tau &\coloneqq \inf\left\{\inf_{\boldsymbol{x}_I \ge \boldsymbol{b}_I} \boldsymbol{x}_I^\top (\Sigma_{II}(t,s))^{-1}\boldsymbol{x}_I \;\middle|\; (t,s) \in \Lambda u^{-2/\alpha}([q,q+1]\times[0,1])\right\} \\ &\ge \tau_1 \coloneqq \inf\left\{\inf_{\boldsymbol{x}_I \ge \boldsymbol{b}_I} \boldsymbol{x}_I^\top (\Sigma_{II}(t,s))^{-1}\boldsymbol{x}_I \;\middle|\; (t,s) \in [0,T]^n\colon |t_l - s_l| > \varepsilon\right\} \\ &> \tau_0 \coloneqq \inf_{\boldsymbol{x}_I \ge \boldsymbol{b}_I} \boldsymbol{x}_I^\top (\Sigma_{II})^{-1}\boldsymbol{x}_I > 0. \end{aligned}$$

Note that the condition $\exists\, l\colon |q_l| > N_u(\varepsilon)$ allows us to separate $\delta(u,\varepsilon) \coloneqq \tau - \tau_0$ from 0 by $\delta(\varepsilon) \coloneqq \tau_1 - \tau_0 > 0$, which depends on ε, but does not depend on u. Since $\tau_0 = \boldsymbol{b}_I^\top(\Sigma_{II})^{-1}\boldsymbol{b}_I = \boldsymbol{b}^\top\Sigma^{-1}\boldsymbol{b}$, we obtain by using the Piterbarg inequality (2.25) the following upper bound:

$$\begin{aligned} &\mathbb{P}\left\{\exists\,(t,s) \in \Lambda u^{-2/\alpha}([q,q+1]\times[0,1])\colon \frac{1}{2}\Big[\boldsymbol{X}(t) + \boldsymbol{X}(s)\Big] > u\boldsymbol{b}\right\} \\ &\qquad \le c_1\, u^{2n/\gamma-1}\, \mathrm{mes}\Big(\Lambda u^{-2/\alpha}([q,q+1]\times[0,1])\Big) \exp\left(-\frac{u^2\tau}{2}\right) \\ &\qquad \le c_2\, \Lambda^{2n}\, u^M \exp\left(-\frac{u^2}{2}\Big[\boldsymbol{b}^\top\Sigma^{-1}\boldsymbol{b} + \delta(\varepsilon)\Big]\right), \end{aligned}$$

which is negligible with respect to $\mathbb{P}\{\boldsymbol{X}(0) > u\boldsymbol{b}\}$ as $u \to \infty$. Summing these bounds, we obtain

$$\limsup_{u\to\infty} \frac{\displaystyle\sum_{\exists\, l\colon |q_l| > N_u(\varepsilon)} P_{\boldsymbol{b}}(q,\Lambda)}{\displaystyle\prod_{l=1}^n u^{-2/\alpha_l}\,\mathbb{P}\left\{\boldsymbol{X}(0) > u\boldsymbol{b}\right\}} = 0.$$

To bound the second sum in (4.27), we divide it further into

$$\sum_{-N_u(\varepsilon)\le q\le N_u(\varepsilon)} P_{\boldsymbol{b}}(q,\Lambda) = \sum_{\substack{-N_u(\varepsilon)\le q\le N_u(\varepsilon)\\ \exists\, l:\ |q_l|=1}} P_{\boldsymbol{b}}(q,\Lambda) + \sum_{\substack{-N_u(\varepsilon)\le q\le N_u(\varepsilon)\\ \forall\, l:\ |q_l|\neq 1}} P_{\boldsymbol{b}}(q,\Lambda) =: A_1 + A_2. \tag{4.28}$$

The probabilities of the second sum can be estimated by Lemma 4.3 as follows:

$$\frac{P_{\boldsymbol{b}}(q,\Lambda)}{\mathbb{P}\left\{\boldsymbol{X}(0) > u\boldsymbol{b}\right\}} \le c\Lambda^{\#\{l:\ k_l=0\}} \prod_{l:\ k_l\neq 0} (|k_l|-1)^{-2} \exp\left(-\frac{1}{4}\,\boldsymbol{w}^\top B_l\,\boldsymbol{w}\,\Lambda^{\alpha_l}\,(|k_l|-1)^{\alpha_l}\right)$$

and therefore

$$\lim_{\Lambda\to\infty}\limsup_{u\to\infty} \frac{A_2}{\mathcal{H}_{\boldsymbol{\alpha},\mathbb{B},\boldsymbol{w}}([0,\Lambda]^n)\prod_{l=1}^n u^{-2/\alpha_l}\mathbb{P}\left\{\boldsymbol{X}(0)>u\boldsymbol{b}\right\}}$$

$$\le c_1 \lim_{\Lambda\to\infty}\sum_l \Lambda^{\#\{l:\ k_l=0\}-n}\exp\left(-\frac{1}{8}\,\boldsymbol{w}^\top B_l\,\boldsymbol{w}\,\Lambda^{\alpha_l}\right) = 0.$$

Next, we show how to bound the first sum. Assume for simplicity that q is such that $|q_l| = 1$ and $|q_{l'}| \neq 1$ for all $l' \neq l$. We have

$$P_{\boldsymbol{b}}(q,\Lambda) = \mathbb{P}\begin{Bmatrix} \forall\, j\neq l\ \exists\, t_j \in \Lambda u^{-2/\alpha_j}[q_j, q_j+1] & & \\ \exists\, t_l \in \Lambda u^{-2/\alpha_l}[1,2] & : & \boldsymbol{X}(t) > u\boldsymbol{b} \\ \exists\, s \in \Lambda u^{-2/\alpha}[0,1] \quad : & & \boldsymbol{X}(s) > u\boldsymbol{b} \end{Bmatrix}$$

$$\le \mathbb{P}\begin{Bmatrix} \forall\, j\neq l\ \exists\, t_j \in \Lambda u^{-2/\alpha_j}[q_j, q_j+1] & & \\ \exists\, t_l \in u^{-2/\alpha_l}\left[\Lambda+\sqrt{\Lambda}, 2\Lambda+\sqrt{\Lambda}\right] & : & \boldsymbol{X}(t) > u\boldsymbol{b} \\ \exists\, s \in \Lambda u^{-2/\alpha}[0,1] \quad : & & \boldsymbol{X}(s) > u\boldsymbol{b} \end{Bmatrix}$$

$$+\, \mathbb{P}\begin{Bmatrix} \exists\, t_j \in \Lambda u^{-2/\alpha_j}[q_j, q_j+1]\ \forall\, j\neq l & \\ \exists\, t_l \in u^{-2/\alpha_l}\left[\Lambda, \Lambda+\sqrt{\Lambda}\right] & :\ \boldsymbol{X}(t) > u\boldsymbol{b} \end{Bmatrix} =: A_3 + A_4.$$

The first probability on the right satisfies the conditions of Lemma 4.3, and therefore

$$\frac{A_3}{\mathbb{P}\left\{\boldsymbol{X}(0) > u\boldsymbol{b}\right\}} \le c_1\Lambda^{\#\{l:\ k_l=0\}} \prod_{l'\neq l,\, k_l\neq 0} \exp\left(-\frac{1}{4}\,\boldsymbol{w}^\top B_l\,\boldsymbol{w}\,\Lambda^{\alpha_l}\,(|q_l|-1)^{\alpha_l}\right)$$

$$\times \exp\left(-\frac{1}{4}\,\boldsymbol{w}^\top B_l\,\boldsymbol{w}\,\Lambda^{\alpha_l/2}\right)$$

Therefore, we obtain

$$\lim_{\Lambda\to\infty}\limsup_{u\to\infty} \frac{\sum_l A_3}{\mathcal{H}_{\boldsymbol{\alpha},\mathbb{B},\boldsymbol{w}}([0,\Lambda]^n)\prod_{l=1}^n u^{-2/\alpha_l}\,\mathbb{P}\left\{\boldsymbol{X}(0)>u\boldsymbol{b}\right\}} = 0.$$

For A_4, we have by Lemma 4.5

$$\frac{A_4}{\mathbb{P}\left\{\boldsymbol{X}(0) > u\boldsymbol{b}\right\}} \sim \mathcal{H}_{\boldsymbol{\alpha},\mathbb{B},\boldsymbol{w}}\left([0,\Lambda] \times \ldots \times \left[0,\sqrt{\Lambda}\right] \times \ldots [0,\Lambda]\right)$$

Consequently, we have

$$\lim_{\Lambda\to\infty} \limsup_{u\to\infty} \frac{\sum\limits_{l} A_4}{T^n \mathcal{H}_{\boldsymbol{\alpha},\mathbb{B},\boldsymbol{w}}\left([0,\Lambda]^n\right) \prod\limits_{l=1}^{n} u^{-2/\alpha_l}\, \mathbb{P}\left\{\boldsymbol{X}(0) > u\boldsymbol{b}\right\}}$$

$$= \lim_{\Lambda\to\infty} \frac{\mathcal{H}_{\boldsymbol{\alpha},\mathbb{B},\boldsymbol{w}}\left([0,\Lambda] \times \ldots \times \left[0,\sqrt{\Lambda}\right] \times \ldots [0,\Lambda]\right)}{\mathcal{H}_{\boldsymbol{\alpha},\mathbb{B},\boldsymbol{w}}\left([0,\Lambda]^n\right)} \leq \lim_{\Lambda\to\infty} \Lambda^{-1/2} = 0.$$

The general case of $q_{\mathcal{I}} \in \{\pm 1\}$ for $\mathcal{I} \subset \{1,\ldots,n\}$ can be addressed similarly.

Positivity of the Pickands constant. To show that the constant is positive we can use the following lower bound:

$$\limsup_{u\to\infty} \frac{\mathcal{H}_{\boldsymbol{\alpha},\mathbb{B},\boldsymbol{w}}\left([0,\Lambda]^n\right)}{\Lambda^n} \geq \liminf_{u\to\infty} \frac{\mathbb{P}\left\{\exists\, t \in [0,T]^n\, \boldsymbol{X}(t) > u\boldsymbol{b}\right\}}{T^n \prod\limits_{l=1}^{n} u^{-2/\alpha_l} \mathbb{P}\left\{\boldsymbol{X}(0) > u\boldsymbol{b}\right\}}$$

$$\geq \liminf_{u\to\infty} \frac{\mathbb{P}\left\{\exists\, t \in [0,\varepsilon]^n \boldsymbol{X}(t) > u\boldsymbol{b}\right\}}{T^n \prod\limits_{l=1}^{n} u^{-2/\alpha_l} \mathbb{P}\left\{\boldsymbol{X}(0) > u\boldsymbol{b}\right\}} \geq \liminf_{u\to\infty} \frac{\widetilde{\Sigma}_1 - \widetilde{\Sigma}_2}{T^n \prod\limits_{l=1}^{n} u^{-2/\alpha_l} \mathbb{P}\left\{\boldsymbol{X}(0) > u\boldsymbol{b}\right\}}, \tag{4.29}$$

where $\widetilde{\Sigma}_1$ and $\widetilde{\Sigma}_2$ are the single and double sum with some Λ' instead of Λ and without odd (in all coordinates) intervals:

$$\widetilde{\Sigma}_1 \coloneqq \sum_{0 \leq k \leq \widetilde{N}_u(\varepsilon)} \mathbb{P}\left\{\exists\, t \in \Lambda' u^{-2/\alpha}[2k, 2k+1] \colon \boldsymbol{X}(t) > u\boldsymbol{b}\right\},$$

$$\widetilde{\Sigma}_2 \coloneqq \sum_{\substack{0 \leq k,\, j \leq \widetilde{N}_u(\varepsilon)\\ k \neq j}} \mathbb{P}\left\{\begin{matrix} \exists\, t \in \Lambda' u^{-2/\alpha}[2k, 2k+1] \colon \boldsymbol{X}(t) > u\boldsymbol{b} \\ \exists\, s \in \Lambda u^{-2/\alpha}[2j, 2j+1] \colon \boldsymbol{X}(s) > u\boldsymbol{b} \end{matrix}\right\}$$

and $\widetilde{N}_u(\varepsilon) = \left\lfloor \varepsilon / 2\Lambda' u^{-2/\alpha} \right\rfloor$. By the same reasoning as above,

$$\liminf_{u\to\infty} \frac{\widetilde{\Sigma}_1}{\prod\limits_{l=1}^{n} u^{-2/\alpha_l} \mathbb{P}\left\{\boldsymbol{X}(0) > u\boldsymbol{b}\right\}} = \left(\frac{\varepsilon}{2}\right)^n \frac{\mathcal{H}_{\boldsymbol{\alpha},\mathbb{B},\boldsymbol{w}}([0,\Lambda'])}{\Lambda'^n},$$

and

$$\limsup_{u\to\infty} \frac{\widetilde{\Sigma}_2}{\prod\limits_{l=1}^{n} u^{-2/\alpha_l} \mathbb{P}\left\{\boldsymbol{X}(0) > u\boldsymbol{b}\right\}} \leq c \left(\frac{\varepsilon}{2}\right)^n \sum_{l} \Lambda'^{\#\{l \colon k_l = 0\} - n} \prod_{k_l \neq 0} \exp\left(-\frac{1}{4}\, \boldsymbol{w}^\top B_l\, \boldsymbol{w}\, \Lambda'^{\alpha_l}\right)$$

Taking Λ' to be large enough, we find that the difference in (4.29) is separated from zero. Hence, its limit is positive. $\square$

4.5.2 Main theorem proof

Proof of Theorem 4.2. We begin the proof by splitting $[0, T]$ into intervals of some small enough $\delta > 0$

$$[0, T]^n = \delta \bigcup_{k \leq N_\delta} [k, k+1], \qquad N_\delta := \left\lceil \frac{T}{\delta} \right\rceil,$$

and applying the Bonferroni inequality, which yields

$$\Sigma_1' - \Sigma_2 \leq \mathbb{P}\left\{ \exists\, t \in [0, T] \colon \boldsymbol{X}(t) > u\boldsymbol{b} \right\} \leq \Sigma_1,$$

where

$$\Sigma_1 := \sum_{k \leq N_\delta} \mathbb{P}\left\{ \exists\, t \in \delta[k, k+1] \colon \boldsymbol{X}(t) > u\boldsymbol{b} \right\}, \qquad \Sigma_2 := \sum_{\substack{k, j \leq N_\delta \\ k \neq j}} \mathbb{P}\left\{ \begin{matrix} \exists\, t \in \delta[k, k+1] \colon \boldsymbol{X}(t) > u\boldsymbol{b} \\ \exists\, s \in \delta[j, j+1] \colon \boldsymbol{X}(s) > u\boldsymbol{b} \end{matrix} \right\}$$

and Σ_1' is defined by the same formula as Σ_1, but with $(N-1)$ instead of N in the upper limit of summation. At this point we split the proof into two parts. First, we will focus on finding the exact asymptotics of the single sum $\Sigma_1 \sim \Sigma_1'$, and then demonstrate that the double sum Σ_2 is negligible with respect to Σ_1.

Single sum. Let min and max applied to a matrix denote component-wise minimum and maximum and let J denote a $d \times d$ matrix of all ones: $J_{kj} = 1$. Take $\varepsilon > 0$ and for each l define two matrices, which bound $B_l(t)$ on $\delta[k, k+1]$ component-wise from below and from above by

$$B_{l,k,\varepsilon,+} := \min_{t \in \delta[k,k+1]} B_t - \varepsilon J, \qquad B_{l,k,\varepsilon,-} := \max_{t \in \delta[k,k+1]} B_t + \varepsilon J.$$

Since for all $t \in [0, T]$ we have $\widetilde{B_t} \rhd 0$ strictly, it follows that $\widetilde{B_{k,\,\varepsilon,\,\pm}} \rhd 0$ if ε is small enough. Denote

$$\mathbb{B}_{k,\varepsilon,\pm} := (B_{l,k,\varepsilon,\pm})_{l=1,\ldots,n}.$$

By Lemma 4.2 the real matrix-valued functions $\mathcal{E}_{\alpha_l,\, B_{l,k,\,\varepsilon,\,\pm}}(s_l)$ are positive definite and give rise to the following bounds on the covariance of $\boldsymbol{X}$:

$$\sum_{l=1}^{n} \mathcal{E}_{\alpha_l,\, B_{l,k,\,\varepsilon,\,-}}(s_l) \leq R(t+s, t) \leq \sum_{l=1}^{n} \mathcal{E}_{\alpha_l,\, B_{l,k,\,\varepsilon,\,+}}(s_l)$$

for small enough s. These functions generate two stationary Gaussian processes $\boldsymbol{Y}_{l,k,\varepsilon,\pm}(s)$, $s \in \mathbb{R}$, which by Lemma 4.4 provide us with bounds on the high excursion probabilities on $\delta[k, k+1]$:

$$\begin{aligned} \mathbb{P}\left\{ \exists\, t \in \delta[k, k+1] \colon \boldsymbol{X}(t) > u\boldsymbol{b} \right\} &\leq \mathbb{P}\left\{ \exists\, t \in \delta[k, k+1] \colon \sum_{l=1}^{n} \boldsymbol{Y}_{l,k,\varepsilon,-}(t) > u\boldsymbol{b} \right\} \\ &\geq \mathbb{P}\left\{ \exists\, t \in \delta[k, k+1] \colon \sum_{l=1}^{n} \boldsymbol{Y}_{l,k,\varepsilon,+}(t) > u\boldsymbol{b} \right\} \end{aligned}$$

Note that the sign plus is on the left and minus is on the right.

Applying Theorem 4.1, we find that

$$\mathbb{P}\left\{ \exists\, t \in \delta[k, k+1] \colon \boldsymbol{Y}_{k,\varepsilon,\pm}(t) > u\boldsymbol{b} \right\} \sim \delta^n\, \mathcal{H}_{\boldsymbol{\alpha},\, \mathbb{B}_{k,\varepsilon,\pm},\, \boldsymbol{w}} \prod_{l=1}^{n} u^{-2/\alpha_l}\, \mathbb{P}\left\{ \boldsymbol{X}(0) > u\boldsymbol{b} \right\}.$$

By adding together all the terms, we obtain

$$\left[\sum_{k=1}^{N_u-1} \mathcal{H}_{\alpha,\mathbb{B}_{k,\varepsilon,+},\boldsymbol{w}}\, \delta^n\right] u^{2/\alpha}\, \mathbb{P}\left\{\boldsymbol{X}(0) > u\,\boldsymbol{b}\right\} \le \Sigma_1'$$

$$\le \Sigma_1 \le \left[\sum_{k=1}^{N_u} \mathcal{H}_{\alpha,\mathbb{B}_{k,\varepsilon,-},\boldsymbol{w}}\, \delta^n\right] u^{2/\alpha}\, \mathbb{P}\left\{\boldsymbol{X}(0) > u\,\boldsymbol{b}\right\}.$$

By continuity of $B \mapsto \mathcal{H}_{\alpha,B,\boldsymbol{b}}$, we have that

$$\lim_{\varepsilon\to 0}\lim_{\delta\to 0}\sum_{k=1}^{N_\delta} \mathcal{H}_{\alpha,\mathbb{B}_{k,\varepsilon,\pm},\boldsymbol{w}}\, \delta^n \xrightarrow[\delta\to 0]{} \int_0^T \mathcal{H}_{\boldsymbol{\alpha},\mathbb{B}(t),\boldsymbol{w}}\, dt\,.$$

Hence, as $u \to \infty$,

$$\lim_{\varepsilon\to 0}\lim_{\delta\to 0} \Sigma_1' \sim \lim_{\varepsilon\to 0}\lim_{\delta\to 0} \Sigma_1 \sim \left[\int_0^T \mathcal{H}_{\boldsymbol{\alpha},\mathbb{B}(t),\boldsymbol{w}}\, dt\right] u^{2/\alpha}\, \mathbb{P}\left\{\boldsymbol{X}(0) > u\,\boldsymbol{b}\right\}.$$

Double sum. The double sum can be estimated by the same argument as in the proof of Theorem 4.1. □

4.6 Appendix

4.6.1 Gordon inequality

The following Slepian-type lemma is stated in [2] for the case where $T \subset \mathbb{R}$, but it can be extended to the following version by standard techniques. Due to its complexity we present it here without proof.

Lemma 4.4 (Gordon inequality)**.** *Let $\boldsymbol{X}(t)$, $t \in T$ and $\boldsymbol{Y}(t)$, $t \in T$ be two centered separable vector-valued Gaussian processes with values in $\mathbb{R}^d$ defined on a separable metric space T. If for all t, $s \in T$ holds*

$$R_{\boldsymbol{X}}(t,t) = R_{\boldsymbol{Y}}(t,t), \qquad R_{\boldsymbol{X}}(t,s) \ge R_{\boldsymbol{Y}}(t,s),$$

then for $\boldsymbol{u} \in \mathbb{R}^d$ holds

$$\mathbb{P}\left\{\exists\, t \in T\colon \boldsymbol{X}(t) > \boldsymbol{u}\right\} \le \mathbb{P}\left\{\exists\, t \in T\colon \boldsymbol{Y}(t) > \boldsymbol{u}\right\}.$$

4.6.2 Local Pickands lemma

The reader may find the uniform multivariate version of the local Pickands lemma in [1]. However, for the needs of this paper this strong result is not necessary, since we obtain uniformity using Gordon's inequality (Lemma 4.4). This is why we present here a simplified version of the local Pickands lemma.

Lemma 4.5. *Let $\boldsymbol{X}(t)$, $t \in [0,T]^n$ be a centered Hölder continuous homogenous Gaussian random field with values in $\mathbb{R}^d$ and covariance R satisfying*

$$\Sigma - R(t) = \sum_{l=1}^{n}\left[B_l |t_l|^{\alpha_l} \mathbb{1}_{t_l \ge 0} + B_l^\top |t_l|^{\alpha_l} \mathbb{1}_{t_l < 0}\right] + \epsilon(t), \qquad \epsilon(t) = o\left(\sum_{l=1}^{n} |t_l|^{\alpha_l}\right) \quad \textit{as} \quad t \to 0,$$

where B_l's are some $d \times d$ real matrices and $\alpha_l \in (0, 2]$. Denote $\alpha \coloneqq (\alpha_l)_{l=1,\dots,n}$, $B \coloneqq (B_l)_{l=1,\dots,n}$ and $\boldsymbol{w} \coloneqq \Sigma^{-1}\widetilde{\boldsymbol{b}}$, where $\widetilde{\boldsymbol{b}}$ is the unique solution of the quadratic programming problem $\Pi_\Sigma(\boldsymbol{b})$. Then the matrix-valued functions $\mathcal{R}_{\alpha_l,B_l} \colon \mathbb{R} \to \mathbb{R}^{d\times d}$ defined by

$$\mathcal{R}_{\alpha_l,B_l}(t_l, s_l) \coloneqq S_{\alpha_l,B_l}(t_l) + S_{\alpha_l,B_l}(-s_l) - S_{\alpha_l,B_l}(t_l - s_l), \qquad S_{\alpha_l,B_l}(t_l) = |t_l|^{\alpha_l}\Big[B_l \mathbb{1}_{t_l \ge 0} + B_l^\top \mathbb{1}_{t_l < 0}\Big]$$

are positive definite and for any $E \subset [0, T]$ containing 0 and closed holds

$$\mathbb{P}\left\{\exists\, t \in u^{-2/\alpha}E \colon \boldsymbol{X}(t) > u\boldsymbol{b}\right\} \sim \mathcal{H}_{\alpha,B,\boldsymbol{w}}(E)\, \mathbb{P}\left\{\boldsymbol{X}(0) > u\boldsymbol{b}\right\}$$

with

$$\mathcal{H}_{\alpha,B,\boldsymbol{w}}(E) = \int_{\mathbb{R}^d} e^{\mathbf{1}^\top \boldsymbol{x}}\, \mathbb{P}\left\{\exists\, t \in E \colon \sum_{l=1}^{n} \mathrm{diag}(\boldsymbol{w})\Big[\boldsymbol{Y}_l(t_l) - S_{\alpha_l,B_l}(t_l)\,\boldsymbol{w}\Big] > \boldsymbol{x}\right\} d\boldsymbol{x} \in (0, \infty),$$

where $\boldsymbol{Y}_l$ is a continuous zero mean Gaussian process associated to the covariance function $\mathcal{R}_{\alpha_l,B_l}(t, s)$.

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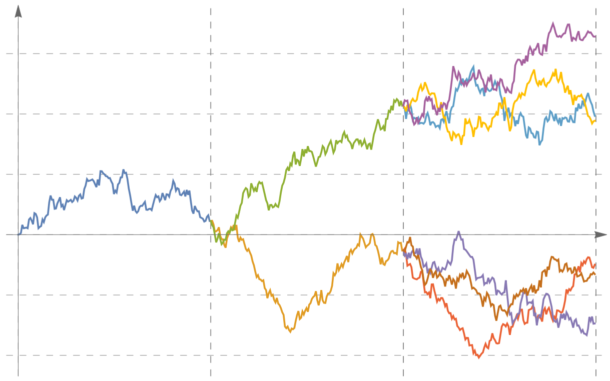
Chapter 5

Extremes of Brownian Decision Trees



A realization of the Brownian decision tree process.

We consider a Brownian motion with linear drift that splits at fixed time points into a fixed number of branches, which may depend on the branching point. For this process, which we shall refer to as the Brownian decision tree, we investigate the exact asymptotics of high exceedance probabilities in finite time horizon, including: the probability that at least one branch exceeds some high threshold, the probability that the largest distance between branches gets large and the probability that all branches simultaneously exceed some high barrier. Additionally, we find the asymptotics for the probability that all branches of at least one of M independent Brownian decision trees exceed a high threshold.

This is a joint work with K.Dębicki and N. Kriukov, submitted to Annals of Applied Probability.

5.1 Introduction

We investigate the exact asymptotics of high exceedance probabilities for the process, which we shall refer to as *the Brownian decision tree.* This process is a close relative of the standard *branching Brownian motion (BBm)* and it can be informally described as follows: at time $t = 0$ a Brownian motion $B(t)$ sets off from zero and runs freely until a non-random time $\tau_1 > 0$, at which it splits into $N_1 \geq 1$ conditionally on the common past independent Brownian motions

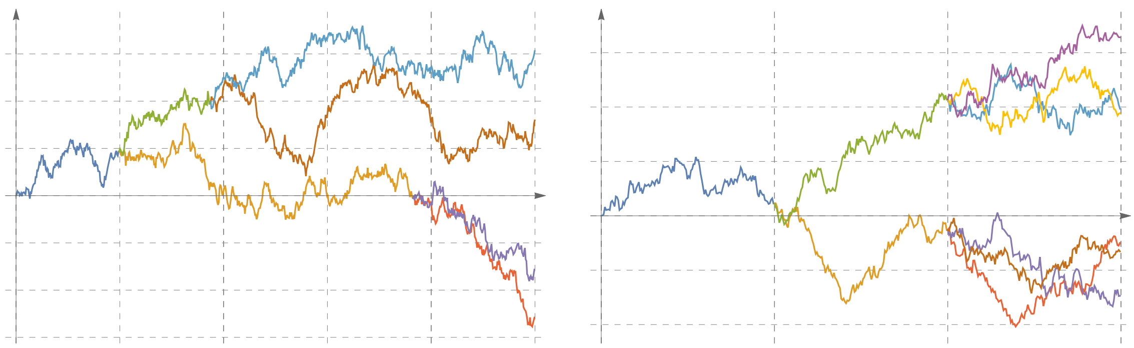
$$B(t) \xmapsto[\text{at } \tau_1]{\text{branching}} \boldsymbol{B}_1(t) = \begin{pmatrix} B(\tau_1) \\ \vdots \\ B(\tau_1) \end{pmatrix} + \boldsymbol{B}_1^*(t - \tau_1),$$

where $\boldsymbol{B}_1^*$ is an $\mathbb{R}^{N_1}$-valued Brownian motion. The resulting vector-valued process again runs freely up to some time point $\tau_2 > \tau_1$, where *each of its components* splits again into $N_2 \geq 1$ particles

$$\boldsymbol{B}_1(t) \xmapsto[\text{at } \tau_2]{\text{branching}} \boldsymbol{B}_2(t) = \begin{pmatrix} \boldsymbol{B}_1(\tau_2) \\ \vdots \\ \boldsymbol{B}_1(\tau_2) \end{pmatrix} + \boldsymbol{B}_2^*(t - \tau_2)$$

and the construction recursively repeats.

There are two differences between this and the classical BBm model as presented, for example, in the seminal paper by Bramson [1]. Firstly, the branching times are non-random, whereas in the standard model the distances between them are exponentially distributed. Secondly, all branches (that is, the components of the vector-valued process described above) undergo splitting into the same number of offsprings and at the same time (in the classical BBm model each branch has its own branching clock). This, along with the usual description of the classical BBm model as a process indexed by a tree, suggests the name *Brownian decision trees*, where the word "decision" refers to the specific type of trees branching at the same points and into the same amount of branches.



(a) Classical branching Brownian motion. (b) Brownian decision tree.

The main findings of the paper are collected in Section 5.3, which we begin with two preliminary results. In Section 5.3.1 we consider the exact asymptotics of the probability that at least one branch of the Brownian decision tree with drift exceeds u, as $u \to \infty$, that is,

$$\mathbb{P}\left\{\exists\, t \in [0, T],\, \exists\, \gamma \in \Gamma \colon B_\gamma(t) - ct > u\right\}, \tag{5.1}$$

where $\Gamma = \{0, \ldots, n-1\}$ is the set of indices of the decision tree branches and $c \in \mathbb{R}$ is a deterministic constant. In Theorem 5.1 we show that (5.1) is asymptotically equal to the product of the number of branches at time T and the probability that a single Brownian motion with drift c crosses level u in time interval $[0,T]$.

A similar approach to the above can be applied for the exact asymptotics of the largest distance between the branches

$$\mathbb{P}\left\{\exists\, t\in[0,T], \exists\, \gamma_1,\gamma_2\in\Gamma\colon B_{\gamma_1}(t)-B_{\gamma_2}(t)>u\right\}, \tag{5.2}$$

as $u\to\infty$, which is derived in Theorem 5.2.

In Section 5.3.3 we focus on the probability

$$\mathbb{P}\left\{\exists\, t\in[0,T]\ \forall\, \gamma\in\Gamma\colon B_{\gamma}(t)-ct>u\right\} \tag{5.3}$$

that for some $t\in[0,T]$ all branches exceed threshold u. This problem is much more complicated and needs a more subtle approach than used in the analysis of (5.1) and (5.2). In order to get the exact asymptotics of (5.3) we develop the technique introduced in [2] for extremes of centered vector-valued Gaussian processes to the branching model considered in this contribution. The main result of this section is displayed in Theorem 5.3, which is supplemented by an extension of Korshunov-Wang inequality [3, 4, 5] (see Proposition 5.7) where a tight upper bound for (5.3), that is valid for all $u>0$, is derived. Complementary to the above findings, we investigate the asymptotics of (5.3) for a version of Brownian decision tree with random numbers of offsprings (Corollary 5.1) and the limiting distribution of the corresponding conditional exceedance times (Corollary 5.2).

In Section 5.3.4 we present asymptotic results for the process which we call *Brownian decision forest*, that consists of a family of M independent Brownian decision trees $\boldsymbol{B}_{\Gamma_i}$, which grow from different points $x_i\in\mathbb{R}$. For this purpose we introduce a partial order on the set of tuples $(\boldsymbol{\tau},\boldsymbol{N},c,x)$, determining the trees of a forest, such that the trees which are high in this order are more likely to exceed the high barrier. In Theorem 5.4 we use this order to find the exact asymptotics of the probability that all branches of at least one tree in a forest exceed some high barrier simultaneously.

We note that in the context of the classical branching Brownian motion most of the asymptotical results focus either on the number of particles or the distribution of the highest branch, which in our setup would be $\max_{\gamma\in\Gamma}B_\gamma(t)$. In both cases the limiting parameter is t approaching infinity. We refer the reader to the classical papers [1, 6, 7, 8, 9]. The two types of questions mentioned above have been investigated extensively in the last decades for various versions of the classical BBm model. Among these versions, the most popular is the Kesten's BBm with absorption (see the original paper [10]). Other models include BBm in random [11, 12] and periodic [13] environments, spatial selection such as in the Brownian bees model [14, 15, 16], spatially-inhomogenous branching rates [17], self-repulsive BBm [18] and many others.

As it turns out, the exact asymptotics of the counterpart of (5.3) for the classical BBm is relatively simpler to obtain than for the analyzed in this contribution Brownian decision trees. Namely, it suffices to observe that the most probable scenario in this case is that the BBm process does not manage to branch even once before hitting the high barrier. Hence, the probability (5.3) is asymptotically equivalent to the probability that the first branching event happens after T times the one-dimensional ruin probability of the Brownian motion. We present here the precise result, the proof of which the reader can find in the Appendix.

Proposition 5.1. *Let $\boldsymbol{B}(t)$, $t\geq 0$ be the classical Branching Brownian motion as described above, with $\tau\sim\mathrm{Exp}(1)$. Then, for any $c\in\mathbb{R}$ we have*

$$\mathbb{P}\left\{\exists\, t\in[0,T]\ \forall\, \gamma\in\Gamma\colon B_{\gamma}(t)-ct>u\right\}\sim e^{-T}\times\sqrt{\frac{2T}{\pi}}\exp\left(-\frac{(u+cT)^2}{2T}\right).$$

The paper is organized as follows. We begin with a construction of the Brownian decision tree process and establish some necessary notation and basic properties in Section 5.2. The main results are presented in Section 5.3. Most of the technical proofs are relegated to Sections 5.4 and 5.5.

5.2 Definitions and basic properties of Brownian decision trees

Let $0 < \tau_1 < \ldots < \tau_\eta < T$ be a finite collection of points, further referred to as *the branching points*, and a sequence of numbers $N_i \geq 1$ with $i = 1, \ldots, \eta$, interpreted as *the numbers of branches generated at* τ_i. Denote by P_i the total number of branches born up to time τ_i, that is,

$$P_i = \prod_{j=1}^{i} N_j,$$

and define the set Γ indexing the branches on $[0, T]$ by

$$\Gamma = \{0, 1, \ldots, P_\eta - 1\}.$$

Let I_i be the $P_i \times P_i$ identity matrix, and $\mathbf{1}_i = (1, \ldots, 1)^\top \in \mathbb{R}^{P_i}$. Denote by $i(t)$ the number of branching points before t

$$i(t) = \max\{i \in \{1, \ldots, \eta\} : t > \tau_i\}.$$

Clearly, each t belongs to the corresponding interval of the form $(\tau_{i(t)}, \tau_{i(t)+1}]$.

Next, take a collection of mutually independent standard Brownian motions $\boldsymbol{B}_i^*(t)$, $t \geq 0$ in $\mathbb{R}^{P_i}$ indexed by $i = 1, \ldots, \eta$ and a one dimensional Brownian motion $B_0^*(t)$, $t \geq 0$. Assume that all these processes are mutually independent. Using the ingredients described above, we construct a new process $\widetilde{\boldsymbol{B}}_\Gamma(t)$, $t \in [0, T]$, which we shall call *the Brownian decision tree*, as follows: for $t \in (\tau_i, \tau_{i+1}]$ set

$$\widetilde{\boldsymbol{B}}_\Gamma(t) := \begin{pmatrix} \widetilde{\boldsymbol{B}}_\Gamma(\tau_i) \\ \vdots \\ \widetilde{\boldsymbol{B}}_\Gamma(\tau_i) \end{pmatrix} + \boldsymbol{B}_i^*(t - \tau_i) \quad \text{for} \quad t \in (\tau_i, \tau_{i+1}]$$

and

$$\widetilde{\boldsymbol{B}}_\Gamma(t) := B_0^*(t) \quad \text{for} \quad t \in [0, \tau_1].$$

Note that $(\widetilde{\boldsymbol{B}}_\Gamma(t))_{t \in (\tau_i, \tau_{i+1}]}$ belongs to $C((\tau_i, \tau_{i+1}], \mathbb{R}^{P_i})$. We can extract the individual branch indexed by $\gamma \in \Gamma$ by taking

$$B_\gamma(t) := \big(\widetilde{\boldsymbol{B}}_\Gamma(t)\big)_{(\gamma \bmod P_{i(t)})+1} \tag{5.4}$$

and denote for two fixed branches γ_1 and γ_2 their *separation moment* at which they diverge by

$$\kappa(\gamma_1, \gamma_2) := \min\{n \in \{1, \ldots, \eta\} : \gamma_1 \neq \gamma_2 \bmod P_n\}.$$

Clearly, $B_{\gamma_1}(t) = B_{\gamma_2}(t)$ for $t \leq \tau_{\kappa(\gamma_1,\gamma_2)}$.

Next, we present some useful properties of Brownian decision trees.

Proposition 5.2. *For $\gamma \in \Gamma$ the process B_γ is a Brownian motion and for $t \in (\tau_1, T]$ it admits the following representation in terms of a collection of independent Brownian motions $\{\boldsymbol{B}_j^*\}_{j=0,\ldots,\eta}$:*

$$B_\gamma(t) = B_0^*(\tau_1) + \sum_{j=1}^{i(t)-1} \big(\boldsymbol{B}_j^*\big)_{(\gamma \bmod P_j)+1}(\tau_{j+1} - \tau_j) + \big(\boldsymbol{B}_{i(t)}^*\big)_{(\gamma \bmod P_{i(t)})+1}(t - \tau_{i(t)}). \tag{5.5}$$

Moreover, for $\gamma_1 \neq \gamma_2$ *and* $t_1,\, t_2 \in [0, T]$ *holds*

$$\mathrm{Cov}(B_{\gamma_1}(t_1), B_{\gamma_2}(t_2)) = \min\{t_1, t_2, \tau_{\kappa(\gamma_1,\gamma_2)}\}. \tag{5.6}$$

The proof of Proposition 5.2 can be found in Appendix.

Sometimes it is more convenient to work with the process $\boldsymbol{B}_\Gamma$, which contains all the branches simultaneously

$$\boldsymbol{B}_\Gamma(t) = (B_\gamma(t))_{\gamma\in\Gamma} \in \mathbb{R}^{P_\eta}. \tag{5.7}$$

The difference between this process and $\widetilde{\boldsymbol{B}}_\Gamma$ is that its values belong to $\mathbb{R}^{P_\eta}$ for each t instead of $\mathbb{R}^{P_{i(t)}}$. The two processes are related as follows:

$$\widetilde{\boldsymbol{B}}_\Gamma(t) = (\boldsymbol{B}_\Gamma(t))_{\gamma\in\{0,\dots,P_{i(t)}-1\}}. \tag{5.8}$$

Unlike $\widetilde{\boldsymbol{B}}_\Gamma(t)$, the variance matrix of this process is degenerate for all $t \in [0, \tau_\eta]$:

$$\det\left(\mathrm{Var}\left(\boldsymbol{B}_\Gamma(t)\right)\right) = 0.$$

Proposition 5.3. *For* $0 \le t_1 < t_2 \le T$, *the random vector* $\boldsymbol{B}_\Gamma(t_2) - \boldsymbol{B}_\Gamma(t_1)$ *is independent of the process* $\boldsymbol{B}_\Gamma(t)|_{t\in[0,t_1]}$.

Let $\Sigma(t)$ denote the covariance matrix of the random vector $\widetilde{\boldsymbol{B}}_\Gamma(t)$:

$$\Sigma(t) \coloneqq \mathbb{E}\left\{\widetilde{\boldsymbol{B}}_\Gamma(t)\,\widetilde{\boldsymbol{B}}_\Gamma^\top(t)\right\}, \qquad t \in [0, T].$$

Proposition 5.4. *For* $t \in (\tau_1, T]$,

$$\Sigma(t) = \begin{pmatrix} \Sigma(\tau_{i(t)}) & \dots & \Sigma(\tau_{i(t)}) \\ \vdots & \ddots & \vdots \\ \Sigma(\tau_{i(t)}) & \dots & \Sigma(\tau_{i(t)}) \end{pmatrix} + (t - \tau_{i(t)}) I_{i(t)},$$

where the first term matrix has $N_{i(t)}^2$ *blocks, all equal to* $\Sigma(\tau_{i(t)})$.

In the next proposition we find the eigenvalues of $\Sigma(t)$.

Proposition 5.5. *For* $t \in [0, T]$, *the eigenvalues of matrix* $\Sigma(t)$ *are given by*

$$\mu_v(t) = (t - \tau_{i(t)}) + \sum_{l=v+1}^{i(t)} (\tau_l - \tau_{l-1}) \prod_{j=l}^{i(t)} N_j, \qquad v = 0, 1, \dots, i(t). \tag{5.9}$$

The multiplicity of $\mu_v(t)$ *equals* $P_v - P_{v-1}$, *except for* μ_0, *the multiplicity of which is* 1. *Additionally, for any* $t \in [0, T]$, *the vector* $\mathbf{1}_{i(t)}$ *is an eigenvector of* $\Sigma(t)$ *corresponding to* $\mu_0(t)$.

Using Proposition 5.5 we can obtain the following result.

Proposition 5.6. *For* $c \in \mathbb{R}$ *and* $T > 0$

$$\mathbb{P}\left\{\boldsymbol{B}_\Gamma(T) - cT\mathbf{1}_\eta > u\mathbf{1}_\eta\right\} \sim u^{-P_\eta} \frac{\mu_0^{P_\eta - 1/2}(T)}{(2\pi)^{P_\eta/2} \prod_{v=1}^{\eta} \mu_v^{(P_v - P_{v-1})/2}(T)} \exp\left(-\frac{(u + cT)^2 P_\eta}{2\mu_0(T)}\right)$$

as $u \to \infty$, *where* $\mu_v(T)$ *are defined in* (5.9).

Remark 5.1. *The result obtained in Proposition 5.6 still hold for* $T = T_u$ *with* $T_u \to T$ *as* $u \to \infty$.

The proofs of Proposition 5.5 and Proposition 5.6 are given in Appendix.

5.3 Main results

In this section we present the main findings of this contribution. We begin with the asymptotic analysis of the high exceedance probability of at least one branch of $\boldsymbol{B}_\Gamma$ with linear drift, then we proceed to the asymptotics of the largest distance between the branches. In Section 5.3.3 we investigate the probability that all branches of $\boldsymbol{B}_\Gamma$ with linear drift exceed high threshold and then extend it to the analysis of a finite collection of independent Brownian decision trees.

5.3.1 High-exeedance of at least one branch

Consider the probability that at least one branch of the branching Brownian decision tree with linear trend exceeds some large threshold u. Clearly, for each $u > 0$ and a standard Brownian motion $B(t), t \in [0, \infty)$

$$\mathbb{P}\left\{\exists\, t \in [0,T],\, \exists\, \gamma \in \Gamma \colon B_\gamma(t) - ct > u\right\} \le P_\eta \mathbb{P}\left\{\exists\, t \in [0,T] \colon B(t) - ct > u\right\},$$

where P_η is the total amount of the branches in the decision tree. It appears that, as $u \to \infty$, the above bound provides the exact asymptotics, as shown in the following theorem.

Theorem 5.1. *For $c \in \mathbb{R}$, as $u \to \infty$,*

$$\begin{aligned}\mathbb{P}\left\{\exists\, t \in [0,T],\, \exists\, \gamma \in \Gamma \colon B_\gamma(t) - ct > u\right\} &\sim P_\eta\, \mathbb{P}\left\{\exists\, t \in [0,T] \colon B(t) - ct > u\right\} \\ &\sim P_\eta \sqrt{\frac{2}{\pi}} \frac{\sqrt{T}}{u + cT} \exp\left(-\frac{(u+cT)^2}{2T}\right).\end{aligned}$$

5.3.2 The largest distance between the branches

Next, we investigate the probability that in time interval $[0, T]$ the largest distance between the branches of the Brownian branching tree with drift gets larger than $u > 0$

$$\mathbb{P}\left\{\exists\, t \in [0,T], \exists\, \gamma_1, \gamma_2 \in \Gamma \colon (B_{\gamma_1}(t) - ct) - (B_{\gamma_2}(t) - ct) > u\right\}.$$

We note that the drifts in the above formula cancel out. Thus in the following result we consider the driftless case.

Theorem 5.2. *As $u \to \infty$*

$$\begin{aligned}&\mathbb{P}\left\{\exists\, t \in [0,T], \exists\, \gamma_1, \gamma_2 \in \Gamma \colon B_{\gamma_1}(t) - B_{\gamma_2}(t) > u\right\} \\ &\qquad \sim P_\eta^2 \frac{N_1 - 1}{N_1} \frac{2\sqrt{T - \tau_1}}{u\sqrt{\pi}} \exp\left(-\frac{u^2}{4(T-\tau_1)}\right).\end{aligned}$$

5.3.3 Simultaneous high-exceedance of all branches

In this section we analyze properties of the simultaneous all-branch high exceedance probability

$$\mathbb{P}\left\{\exists\, t \in [0,T]\ \forall \gamma \in \Gamma \colon B_\gamma(t) - ct > u\right\}. \tag{5.10}$$

We begin with a non-asymptotic result. Obviously, for each $u > 0$

$$\mathbb{P}\left\{\exists\, t \in [0,T]\ \forall \gamma \in \Gamma \colon B_\gamma(t) - ct > u\right\} \ge \mathbb{P}\left\{\forall \gamma \in \Gamma \colon B_\gamma(T) - cT > u\right\}, \tag{5.11}$$

where the asymptotics, as $u \to \infty$, of the probability on the righthand side of the above inequality is given in Proposition 5.6.

In the following proposition we find an upper bound for (5.10) which differs from the bound (5.11) by some constant.

Proposition 5.7. *For $c \in \mathbb{R}$ and any $u > 0$,*

$$\mathbb{P}\left\{\exists\, t \in [0, T]\ \forall \gamma \in \Gamma \colon B_\gamma(t) - ct > u\right\} \leq C\, \mathbb{P}\left\{\forall \gamma \in \Gamma \colon B_\gamma(T) - cT > u\right\},$$

where $C = \left(\mathbb{P}\left\{\xi > |c|\right\}\right)^{-P_\eta}$ with ξ a standard normal random variable.

In order to present the exact asymptotics of (5.10) as $u \to \infty$ we need to introduce the following constant

$$\mathcal{H}_{\mathcal{N},\lambda} := \int_{\mathbb{R}^{\mathcal{N}}} \mathbb{P}\left\{\exists\, t \in (0, \infty),\, \forall\, i \in \{1, \ldots, \mathcal{N}\} \colon \lambda B_i^*(t) - \lambda^2 t > x_i\right\} e^{\sum_{i=1}^{\mathcal{N}} x_i} \mathrm{d}\boldsymbol{x}, \tag{5.12}$$

where $B_i^*(t),\ t \in [0, \infty),\ i = 1, ..., \mathcal{N} \in \mathbb{N}$ are mutually independent standard Brownian motions and $\lambda > 0$. The finiteness of this constant is shown in Lemma 5.4.

The following result constitutes the main finding of this section.

Theorem 5.3. *For $c \in \mathbb{R}$, as $u \to \infty$,*

$$\mathbb{P}\left\{\exists\, t \in [0, T]\ \forall\, \gamma \in \Gamma \colon B_\gamma(t) - ct > u\right\} \sim \mathcal{H}_{P_\eta, 1/\mu_0(T)}\, \mathbb{P}\left\{\forall\, \gamma \in \Gamma \colon B_\gamma(T) - cT > u\right\}.$$

Remark 5.2. *The results obtained in Proposition 5.7 and Theorem 5.3 still hold for $T = T_u$ with $T_u \to T$ as $u \to \infty$.*

Interestingly, the above result can be extended to the version of Brownian decision tree with random numbers of offsprings N_i. For a random vector $\boldsymbol{N}$, let $\mathrm{essinf}(\boldsymbol{N})$ denote the componentwise essential infimum of $\boldsymbol{N}$.

Corollary 5.1. *Assume that $N_i \in \mathbb{N}$ are independent random variables. Then,*

$$\mathbb{P}\left\{\exists\, t \in [0, T]\ \forall \gamma \in \Gamma \colon\ B_\gamma(t) - ct > u\right\} \sim \mathcal{H}_{P_\eta, 1/\mu_0(T)} \prod_{i=1}^{\eta} \mathbb{P}\left\{N_i = \mathrm{essinf}(N_i)\right\}$$
$$\times\, \mathbb{P}\left\{\forall\, \gamma \in \Gamma \colon B_\gamma(T) - ct > u\ \mid \boldsymbol{N} = \mathrm{essinf}(\boldsymbol{N})\right\},$$

as $u \to \infty$. The constant $\mathcal{H}_{P_\eta, 1/\mu_0(T)}$ is calculated under the assumption that $\boldsymbol{N} = \mathrm{essinf}(\boldsymbol{N})$.

The results obtained in Theorem 5.3 and Proposition 5.6 allow us to derive the asymptotics for the conditional first time of simultaneous exceedance of all branches

$$\mathcal{T}(u) := \inf\{t \in [0, T] \colon \forall\, \gamma \in \Gamma \colon B_\gamma(t) - ct > u\}.$$

Corollary 5.2. *Then for any $x, y \in (0, +\infty)$ such that $x > y$, holds*

$$\lim_{u \to \infty} \mathbb{P}\left\{u^2(T - \mathcal{T}(u)) \geq x \mid \mathcal{T}(u) \leq T - y/u^2\right\} = \exp\left(-\frac{(x-y)P_\eta}{2\mu_0^2(T)}\right).$$

Example 5.1 (Binary tree)**.** *Suppose that the difference between branching points equals one (i.e. $\tau_i - \tau_{i-1} = 1$ for all $i \in \mathbb{N}$) and each process always splits into two (i.e. $N_i = 2$ for all $i \in \mathbb{N}$). For the sake of simplicity, let T be integer. Then*

$$\mu_v(T) = 2^{T-v} - 1.$$

Combining Theorem 5.3 and Proposition 5.6 we obtain that as $u \to \infty$

$$\mathbb{P}\left\{\exists\, t \in [0,T]\ \forall \gamma \in \Gamma \colon B_\gamma(t) - ct > u\right\}$$

$$\sim \frac{\mathcal{H}_{2^{T-1},1/(2^T-1)}(2^T-1)^{2^{T-1}-1/2}}{(2\pi)^{2^{T-2}}\prod_{v=1}^{T-1}(2^{T-v}-1)^{2^{v-2}}}u^{-2^{T-1}}\exp\left(-\frac{2^{T-2}}{2^T-1}(u+cT)^2\right).$$

5.3.4 Brownian decision forests

In this section we shall consider a set of $M \in \mathbb{N}$ independent Brownian decision trees with drift. That is, for $i = 1, \ldots, M$ we take a triple $(\boldsymbol{\tau}_i, \boldsymbol{N}_i, c)$, where $\boldsymbol{\tau}_i$ and $\boldsymbol{N}_i$ are two vectors and c_i is a constant, associate to each of them a Brownian decision tree $\boldsymbol{B}_{\Gamma_i}$ as described in Section 5.2 (such that all of them are independent), and set

$$\boldsymbol{W}_i(t) \coloneqq \boldsymbol{B}_{\Gamma_i}(t) + (x_i - c_i\, t)\, \mathbf{1}_{\eta_i},$$

where x_i's are interpreted as the starting points of these trees. Each tree $\boldsymbol{W}_i(t)$ is thus uniquely defined by its tuple $(\boldsymbol{\tau}_i, \boldsymbol{N}_i, c_i, x_i)$. Let us define a partial order relation $\succcurlyeq$ on the set of trees as follows: we write $(\boldsymbol{\tau}_1, \boldsymbol{N}_1, c_1, x_1) \succcurlyeq (\boldsymbol{\tau}_2, \boldsymbol{N}_2, c_2, x_2)$ if one of the following three conditions holds:

(i) $$\frac{\mu_{0,1}(T)}{P_{\eta_1}} > \frac{\mu_{0,2}(T)}{P_{\eta_2}},$$

(ii) $$\frac{\mu_{0,1}(T)}{P_{\eta_1}} = \frac{\mu_{0,2}(T)}{P_{\eta_2}} \quad \text{and} \quad c_1T - x_1 < c_2T - x_2\,,$$

(iii) $$\frac{\mu_{0,1}(T)}{P_{\eta_1}} = \frac{\mu_{0,2}(T)}{P_{\eta_2}} \quad \text{and} \quad c_1T - x_1 = c_2T - x_2 \quad \text{and} \quad P_{\eta_1} \le P_{\eta_2}.$$

In other words, it is the lexicographic order on the tuples $(\mu_0(T)/P_\eta,\ x - cT,\ -P_\eta)$. One may notice that this order is full. Next, let

$$(\boldsymbol{\tau}_1, \boldsymbol{N}_1, c_1, x_1) \approx (\boldsymbol{\tau}_2, \boldsymbol{N}_2, c_2, x_2) \iff \begin{cases}(\boldsymbol{\tau}_1, \boldsymbol{N}_1, c_1, x_1) \preccurlyeq (\boldsymbol{\tau}_2, \boldsymbol{N}_2, c_2, x_2),\\ (\boldsymbol{\tau}_1, \boldsymbol{N}_1, c_1, x_1) \succcurlyeq (\boldsymbol{\tau}_2, \boldsymbol{N}_2, c_2, x_2)\end{cases}$$

and

$$(\boldsymbol{\tau}_1, \boldsymbol{N}_1, c_1, x_1) \succ (\boldsymbol{\tau}_2, \boldsymbol{N}_2, c_2, x_2) \iff \begin{cases}(\boldsymbol{\tau}_1, \boldsymbol{N}_1, c_1, x_1) \succcurlyeq (\boldsymbol{\tau}_2, \boldsymbol{N}_2, c_2, x_2),\\ (\boldsymbol{\tau}_1, \boldsymbol{N}_1, c_1, x_1) \not\approx (\boldsymbol{\tau}_2, \boldsymbol{N}_2, c_2, x_2).\end{cases}$$

Combining Theorem 5.3 and Proposition 5.6 we straightforwardly obtain the following result.

Lemma 5.1. *The following equivalences hold:*

$$(\boldsymbol{\tau}_i, \boldsymbol{N}_i, c_i, x_i) \succ (\boldsymbol{\tau}_j, \boldsymbol{N}_j, c_j, x_j) \iff \lim_{u\to\infty}\frac{\mathbb{P}\left\{\exists t \in [0,T]:\ \boldsymbol{W}_i(t) > u\mathbf{1}_{\eta_i}\right\}}{\mathbb{P}\left\{\exists t \in [0,T]:\ \boldsymbol{W}_j(t) > u\mathbf{1}_{\eta_j}\right\}} = \infty,$$

$$(\boldsymbol{\tau}_i, \boldsymbol{N}_i, c_i, x_i) \prec (\boldsymbol{\tau}_j, \boldsymbol{N}_j, c_j, x_j) \iff \lim_{u\to\infty}\frac{\mathbb{P}\left\{\exists t \in [0,T]:\ \boldsymbol{W}_i(t) > u\mathbf{1}_{\eta_i}\right\}}{\mathbb{P}\left\{\exists t \in [0,T]:\ \boldsymbol{W}_j(t) > u\mathbf{1}_{\eta_j}\right\}} = 0,$$

$$(\boldsymbol{\tau}_i, \boldsymbol{N}_i, c_i, x_i) \approx (\boldsymbol{\tau}_j, \boldsymbol{N}_j, c_j, x_j) \iff \lim_{u\to\infty}\frac{\mathbb{P}\left\{\exists t \in [0,T]:\ \boldsymbol{W}_i(t) > u\mathbf{1}_{\eta_i}\right\}}{\mathbb{P}\left\{\exists t \in [0,T]:\ \boldsymbol{W}_j(t) > u\mathbf{1}_{\eta_j}\right\}} \in (0,\infty).$$

Lemma 5.1 allows us to find the asymptotics for the probability that all branches of at least one of our M trees exceed the threshold u.

Theorem 5.4. *Let us define by A the set of indices $i \in \{1, \dots, M\}$ for which the corresponding set $(\boldsymbol{\tau}_i, \boldsymbol{N}_i, c_i, x_i)$ is the maximal among all given. Then, as $u \to \infty$,*

$$\mathbb{P}\left\{\exists\, i \in \{1, \dots, M\}, t \in [0,T] \colon \boldsymbol{W}_i(t) > u\mathbf{1}_{\eta_i}\right\} \sim \sum_{i \in A} \mathbb{P}\left\{\exists\, t \in [0,T] \colon \boldsymbol{W}_i(t) > u\mathbf{1}_{\eta_i}\right\},$$

where the asymptotics of each term is given in Theorem 5.3.

5.4 Proofs

5.4.1 Proof of Proposition 5.1

Let $\tau \sim \mathrm{Exp}(1)$ be the first branching moment. Define by $\mathfrak{d}$ the dimension of vector $\boldsymbol{B}(t)$ (i.e. the amount of branches at the point t), and $\mathbf{1}(t) = (1, \dots, 1) \in \mathbb{R}^{\mathfrak{d}}$. Then,

$$\begin{aligned}
&\mathbb{P}\left\{\exists\, t \in [0,T] \colon \boldsymbol{B}(t) - ct\mathbf{1}(t) > u\mathbf{1}(t),\ \tau \geq T\right\} \\
&\qquad \leq \mathbb{P}\left\{\exists\, t \in [0,T] \colon \boldsymbol{B}(t) - ct\mathbf{1}(t) > u\mathbf{1}(t)\right\} \\
&\qquad \leq \mathbb{P}\left\{\exists\, t \in [0,T] \colon \boldsymbol{B}(t) - ct\mathbf{1}(t) > u\mathbf{1}(t),\ \tau \leq T - 2u^{-1}\right\} \\
&\qquad\qquad + \mathbb{P}\left\{\exists\, t \in [0,T] \colon \boldsymbol{B}(t) - ct\mathbf{1}(t) > u\mathbf{1}(t),\ \tau \in [T - 2u^{-1}, T]\right\} \\
&\qquad\qquad + \mathbb{P}\left\{\exists t\, \in [0,T] \colon \boldsymbol{B}(t) - ct\mathbf{1}(t) > u\mathbf{1}(t),\ \tau \geq T\right\} \\
&\qquad =: S_1 + S_2 + S_3
\end{aligned}$$

Consider each term separately. For S_3, using that $(\boldsymbol{B} \mid \tau \geq T)$ for $t \in [0,T]$ is a Brownian motion independent of τ, we have

$$\begin{aligned}
S_3 &= \mathbb{P}\left\{\tau \geq T\right\} \mathbb{P}\left\{\exists\, t \in [0,T] \colon \boldsymbol{B}(t) - ct\mathbf{1}(t) > u\mathbf{1}(t) \mid \tau \geq T\right\} \\
&\sim e^{-T}\sqrt{\frac{2T}{\pi}}\, u^{-1} \exp\left(-\frac{(u+cT)^2}{2T}\right)
\end{aligned}$$

as $u \to \infty$. Considering S_2, again using that B_1 is Brownian motion independent of τ, as $u \to \infty$

$$\begin{aligned}
S_2 &\leq \mathbb{P}\left\{\tau \in [T - 2u^{-1}, T]\right\} \mathbb{P}\left\{\exists\, t \in [0,T] \colon B_1(t) - ct > u\right\} \\
&\sim (e^{2u^{-1}} - 1)\, e^{-T} \sqrt{\frac{2T}{\pi}}\, u^{-1} \exp\left(-\frac{(u+cT)^2}{2T}\right),
\end{aligned}$$

implies that

$$\lim_{u \to \infty} \frac{S_2}{S_3} \to 0.$$

Then, for S_1

$$\begin{aligned}
S_1 &\leq \mathbb{P}\left\{\exists\, t \in [0, T - u^{-1}] \colon B_1(t) - ct > u,\ \tau \leq T - 2u^{-1}\right\} \\
&\quad + \mathbb{P}\left\{\exists\, t \in [T - u^{-1}, T] \colon \frac{B_1(t) + B_2(t)}{2} - ct > u,\ \tau \leq T - 2u^{-1}\right\} =: Z_1 + Z_2
\end{aligned}$$

For Z_1, using that

$$Z_1 \sim (1 - e^{-T})\sqrt{\frac{2T}{\pi}}\, u^{-1} \exp\left(-\frac{(u + c(T - u^{-1}))^2}{2(T - u^{-1})}\right),$$

we obtain

$$\lim_{u\to\infty}\frac{Z_1}{S_3}=\lim_{u\to\infty}\frac{1-e^{-T}}{e^{-T}}\exp\left(\frac{(u+cT)^2}{2T}-\frac{(u+c(T-u^{-1}))^2}{2(T-u^{-1})}\right)$$
$$=\lim_{u\to\infty}\frac{1-e^{-T}}{e^{-T}}\exp\left(\frac{-u^{-1}(u+cT)^2-4cT^2+2Tc^2u^{-2}}{2T(T-u^{-1})}\right)=0.$$

Finally,

$$Z_2=\mathbb{E}\left\{\mathbb{I}_{\{\tau<T-2u^{-1}\}}\,\mathbb{P}\left\{\exists\, t\in[T-u^{-1},T]\colon \frac{B_1(t)+B_2(t)}{2}-ct>u\;\middle|\;\tau\right\}\right\}$$
$$\le\sup_{\mathfrak{t}\in[0,T-2u^{-1}]}\mathbb{P}\left\{\exists\, t\in[T-u^{-1},T]\colon \frac{B_1(t)+B_2(t)}{2}>u-|c|\,T\;\middle|\;\tau=\mathfrak{t}\right\}.$$

Since $(B_1(t)+B_2(t)\mid\tau=\mathfrak{t})$ is a Gaussian process and for any $\mathfrak{t}\in[0,T-u^{-1}]$

$$\operatorname{Var}\left(\frac{B_1(t)+B_2(t)}{2}\;\middle|\;\tau=\mathfrak{t}\right)=\begin{cases}\frac{t+\mathfrak{t}}{2}, & t>\mathfrak{t},\\ t, & t\le\mathfrak{t},\end{cases}$$

we can apply Piterbarg inequality (see, e.g., [19, Theorem 8.1]) obtaining that for $u>|c|\,T$

$$\mathbb{P}\left\{\exists\, t\in[T-u^{-1},T]\colon \frac{B_1(t)+B_2(t)}{2}>u-|c|\,T\;\middle|\;\tau=\mathfrak{t}\right\}$$
$$\le C(u-|c|\,T)^{\alpha}\exp\left(-\frac{(u-|c|\,T)^2}{2\sigma_{\mathfrak{t}}}\right),$$

where

$$\sigma_{\mathfrak{t}}=\max_{t\in[0,T]}\operatorname{Var}\left(\frac{B_1(t)+B_2(t)}{2}\right)=\frac{T+\mathfrak{t}}{2}$$

and $C>0$, $\alpha\in\mathbb{R}$ are some constants. Using that C, α does not depend ot $\mathfrak{t}$, we get

$$Z_2\le Cu^{\alpha}\sup_{\mathfrak{t}\in[0,T-2u^{-1}]}\exp\left(-\frac{(u-|c|\,T)^2}{2\sigma_{\mathfrak{t}}}\right)=Cu^{\alpha}\exp\left(-\frac{(u-|c|\,T)^2}{2(T-u^{-1})}\right).$$

It remains to note that $Z_2/S_3\to 0$ as $u\to\infty$ by the same reason as Z_1/S_3.

5.4.2 Proof ot Theorem 5.1

We begin with the observation that

$$\mathbb{P}\left\{\exists\, t\in[0,T],\gamma\in\Gamma\colon B_\gamma(t)-ct>u\right\}$$
$$\ge\sum_{\gamma\in\Gamma}\mathbb{P}\left\{\exists\, t\in[0,T]\colon B_\gamma(t)-ct>u\right\}-\sum_{\substack{\gamma_1,\gamma_2\in\Gamma\\ \gamma_1\neq\gamma_2}}\mathbb{P}\left\{\exists\, t\in[0,T]\colon \begin{array}{r}B_{\gamma_1}(t)-ct>u,\\ B_{\gamma_2}(t)-ct>u\end{array}\right\}$$

and

$$\mathbb{P}\left\{\exists\, t\in[0,T],\gamma\in\Gamma\colon B_\gamma(t)-ct>u\right\}\le\sum_{\gamma\in\Gamma}\mathbb{P}\left\{\exists\, t\in[0,T]\colon B_\gamma(t)-ct>u\right\},$$

where branch B_γ is a Brownian motion. By [20, Formula 1.1.4 and Appendix 2, Section 8], we have

$$\mathbb{P}\left\{\exists\, t \in [0,T]\colon B_\gamma(t) - ct > u\right\} \sim \frac{2\sqrt{T}}{\sqrt{2\pi}(u+cT)} \exp\left(-\frac{(u+cT)^2}{2T}\right) \tag{5.13}$$

as $u \to \infty$. For the double sum on the left hand side, we have the following upper bound

$$\mathbb{P}\left\{\exists\, t \in [0,T]\colon \begin{array}{r} B_{\gamma_1}(t) - ct > u, \\ B_{\gamma_2}(t) - ct > u \end{array}\right\} \le C\,\mathbb{P}\left\{\begin{array}{r} B_{\gamma_1}(T) - cT > u, \\ B_{\gamma_2}(T) - cT > u \end{array}\right\}.$$

Here C is some positive constant independent of u; see [21, Theorem 3.1]. Since $(B_{\gamma_1}(T), B_{\gamma_2}(T))$ is a Gaussian vector with covariance matrix

$$\Sigma = \begin{pmatrix} T & \tau_{\kappa(\gamma_1,\gamma_2)} \\ \tau_{\kappa(\gamma_1,\gamma_2)} & T \end{pmatrix}$$

we have for some positive constant $C_2 > 0$ as $u \to \infty$

$$\mathbb{P}\left\{B_{\gamma_1}(T) + cT > u,\, B_{\gamma_2}(T) - cT > u\right\} \sim C_2\, u^{-2} \exp\left(-\frac{(u+cT)^2}{2\left(T + \tau_{\kappa(\gamma_1,\gamma_2)}\right)}\right).$$

Using that $\tau_{\kappa(\gamma_1,\gamma_2)} < T$ for $\gamma_1 \neq \gamma_2$, we obtain by (5.13)

$$\mathbb{P}\left\{\exists\, t \in [0,T]\colon \begin{array}{r} B_{\gamma_1}(t) - ct > u, \\ B_{\gamma_2}(t) - ct > u \end{array}\right\} = o\left(\mathbb{P}\left\{\exists\, t \in [0,T]\colon B_\gamma(t) - ct > u\right\}\right)$$

for any $\gamma, \gamma_1, \gamma_2 \in \Gamma$, $\gamma_1 \neq \gamma_2$, establishing the claim.

5.4.3 Proof of Theorem 5.2

Note that

$$\begin{aligned}
&\sum_{\gamma_1,\gamma_2\in\Gamma} \mathbb{P}\left\{\exists\, t \in [0,T]\colon\ B_{\gamma_1}(t) - B_{\gamma_2}(t) > u\right\} \\
&\quad\ge \mathbb{P}\left\{\exists\, t \in [0,T], \exists\, \gamma_1,\gamma_2 \in \Gamma\colon\ B_{\gamma_1}(t) - B_{\gamma_2}(t) > u\right\} \\
&\quad\ge \sum_{\gamma_1,\gamma_2\in\Gamma} \mathbb{P}\left\{\exists\, t \in [0,T]\colon\ B_{\gamma_1}(t) - B_{\gamma_2}(t) > u\right\} \\
&\qquad - \sum_{\substack{\gamma_1,\gamma_2,\gamma_3,\gamma_4\in\Gamma \\ \{\gamma_1,\gamma_2\}\neq\{\gamma_3,\gamma_4\}}} \mathbb{P}\left\{\exists\, t,s \in [0,T]\colon B_{\gamma_1}(t) - B_{\gamma_2}(t) > u, B_{\gamma_3}(s) - B_{\gamma_4}(s) > u\right\}.
\end{aligned}$$

For any $\gamma_1 \neq \gamma_2$, the process $B_{\gamma_1} - B_{\gamma_2}$ admits the following representation

$$B_{\gamma_1}(t) - B_{\gamma_2}(t) = \begin{cases} 0, & t \le \tau_{\kappa(\gamma_1,\gamma_2)}, \\ B^*(2(t - \tau_{\kappa(\gamma_1,\gamma_2)}), & t > \tau_{\kappa(\gamma_1,\gamma_2)}, \end{cases}$$

where $B^*(t)$ is Brownian motion. Hence, as $u \to \infty$,

$$\begin{aligned}
&\mathbb{P}\left\{\exists\, t \in [0,T]\colon B_{\gamma_1}(t) - B_{\gamma_2} > u\right\} \\
&\qquad = \mathbb{P}\left\{\exists\, t \in [\tau_{\kappa(\gamma_1,\gamma_2)}, T]\colon B^*(2(t - \tau_{\kappa(\gamma_1-\gamma_2)})) > u\right\}
\end{aligned}$$

$$= \mathbb{P}\left\{\exists\, t \in [0, 2(T - \tau_{\kappa(\gamma_1,\gamma_2)})] \colon B^*(t) > u\right\}$$

$$\sim \frac{2\sqrt{2(T - \tau_{\kappa(\gamma_1,\gamma_2)})}}{u\sqrt{2\pi}} \exp\left(-\frac{u^2}{4(T - \tau_{\kappa(\gamma_1,\gamma_2)})}\right).$$

For $\gamma_1, \gamma_2, \gamma_3, \gamma_4 \in \Gamma$, if $\kappa(\gamma_1, \gamma_2) < \kappa(\gamma_3, \gamma_4)$, then as $u \to \infty$

$$\frac{\mathbb{P}\left\{\exists\, t \in [0, T] \colon B_{\gamma_3}(t) - B_{\gamma_4} > u\right\}}{\mathbb{P}\left\{\exists\, t \in [0, T] \colon B_{\gamma_1}(t) - B_{\gamma_2} > u\right\}} \to 0.$$

The latter implies that

$$\sum_{\gamma_1,\gamma_2 \in \Gamma} \mathbb{P}\left\{\exists\, t \in [0, T] \colon B_{\gamma_1}(t) - B_{\gamma_2}(t) > u\right\}$$

$$\sim \sum_{\substack{\gamma_1,\gamma_2 \in \Gamma \\ \kappa(\gamma_1,\gamma_2)=1}} \frac{2\sqrt{2(T - \tau_1)}}{u\sqrt{2\pi}} \exp\left(-\frac{u^2}{4(T - \tau_1)}\right)$$

$$= N_1(N_1 - 1)\left(\frac{P_\eta}{N_1}\right)^2 \frac{2\sqrt{2(T - \tau_1)}}{u\sqrt{2\pi}} \exp\left(-\frac{u^2}{4(T - \tau_1)}\right).$$

Next, consider the double events' probabilities. For $\gamma_1, \gamma_2, \gamma_3, \gamma_4 \in \Gamma$,

$$\mathbb{P}\left\{\exists\, t, s \in [0, T] \colon B_{\gamma_1}(t) - B_{\gamma_2}(t) > u, B_{\gamma_3}(s) - B_{\gamma_4}(s) > u\right\}$$

$$\leq \mathbb{P}\left\{\exists\, t, s \in [0, T] \colon \frac{B_{\gamma_1}(t) - B_{\gamma_2}(t) + B_{\gamma_3}(s) - B_{\gamma_4}(s)}{2} > u\right\}.$$

Using that $(B_{\gamma_1}(t) - B_{\gamma_2}(t) + B_{\gamma_3}(s) - B_{\gamma_4}(s))/2$ is a Gaussian random field and

$$\operatorname{Var}\left(\frac{B_{\gamma_1}(t) - B_{\gamma_2}(t) + B_{\gamma_3}(s) - B_{\gamma_4}(s)}{2}\right) \leq \frac{3T + \tau_\eta - 4\tau_1}{2},$$

by Piterbarg inequality (see, e.g., [19, Theorem 8.1]) there exist some constants $C > 0,\ \alpha \in \mathbb{R}$

$$\mathbb{P}\left\{\exists\, t, s \in [0, T] \colon B_{\gamma_1}(t) - B_{\gamma_2}(t) + B_{\gamma_3}(s) - B_{\gamma_4}(s) > 2u\right\}$$

$$\leq C u^\alpha \exp\left(-\frac{u^2}{3T + \tau_\eta - 4\tau_1}\right).$$

The above inequality implies that

$$\frac{\displaystyle\sum_{\substack{\gamma_1,\gamma_2,\gamma_3,\gamma_4 \in \Gamma \\ \{\gamma_1,\gamma_2\} \neq \{\gamma_3,\gamma_4\}}} \mathbb{P}\left\{\exists\, t, s \in [0, T] \colon \begin{matrix} B_{\gamma_1}(t) - B_{\gamma_2}(t) > u, \\ B_{\gamma_3}(s) - B_{\gamma_4}(s) > u \end{matrix}\right\}}{\displaystyle\sum_{\gamma_1,\gamma_2 \in \Gamma} \mathbb{P}\left\{\exists t \in [0, T] \,\colon\, B_{\gamma_1}(t) - B_{\gamma_2}(t) > u\right\}} \to 0$$

as $u \to \infty$, establishing the proof.

5.4.4 Proofs of results on simultaneous high-exceedance probability of all branches

Until the end of this secion, let us define

$$\boldsymbol{W}(t) = \boldsymbol{B}_\Gamma(t) - \boldsymbol{c}t, \qquad \boldsymbol{c} = c\,\mathbf{1}_\eta, \tag{5.14}$$

where $c \in \mathbb{R}$ is fixed. Then the the all-branch high exceedance probability can be stated as

$$\mathbb{P}\left\{\exists\, t \in [0,T],\, \forall \gamma \in \Gamma\colon\; B_\gamma(t) - ct > u\right\} = \mathbb{P}\left\{\exists\, t \in [0,T]\colon \boldsymbol{W}(t) > u\mathbf{1}_\eta\right\}.$$

Proof of Proposition 5.7. As in the proof of [3, Theorem 1.1] define a stopping time by

$$\mathcal{T} = \inf\{t \in [0,T]\colon \boldsymbol{W}(t) > u\mathbf{1}_\eta\} = \inf\{t \in [0,T]\colon \forall \gamma \in \Gamma\; B_\gamma(t) - ct > u\},$$

using the convention $\inf \varnothing = T + 1$ and write $F_\mathcal{T}$ for its distribution function. Since the paths of $\boldsymbol{W}$ are continuous, $\mathcal{T}$ is well-defined. By Proposition 5.3,

$$\begin{aligned}
\mathbb{P}\left\{\boldsymbol{W}(T) > u\mathbf{1}_\eta\right\} &= \int_0^T dF_\mathcal{T}(t)\, \mathbb{P}\left\{\boldsymbol{W}(T) > u\mathbf{1}_\eta \mid \mathcal{T} = t\right\} \\
&= \int_0^T dF_\mathcal{T}(t)\, \mathbb{P}\left\{\forall \gamma \in \Gamma\colon B_\gamma(T) - cT > u \mid \mathcal{T} = t\right\} \\
&\geq \int_0^T dF_\mathcal{T}(t)\, \mathbb{P}\left\{\forall \gamma \in \Gamma\colon B_\gamma(T) - B_\gamma(t) > c(T-t)\right\}.
\end{aligned}$$

Consider the last probability. Define for $t \in [0,T_1]$ and $\gamma \in \Gamma$ a Gaussian random variable $\overline{B}_\gamma(t)$ by

$$\overline{B}_\gamma(t) = \left(B_\gamma(T) - B_\gamma(t)\right).$$

Its variance is

$$\operatorname{Var}\left(\overline{B}_\gamma(t)\right) = \operatorname{Var}\left(B_\gamma(T) - B_\gamma(t)\right) = T_1 - t,$$

and the covariance between them

$$\operatorname{Cov}\left(\overline{B}_{\gamma_1}(t), \overline{B}_{\gamma_2}(t)\right) = \min\{T, \tau_{\kappa(\gamma_1,\gamma_2)}\} - \min\{t, \tau_{\kappa(\gamma_1,\gamma_2)}\} \geq 0.$$

Hence, using Slepian inequality (see e.g., Theorem 2.2.1 in [22]) we arrive at

$$\begin{aligned}
\mathbb{P}\left\{\forall\, \gamma \in \Gamma\colon \overline{B}_\gamma(t) > c(T_1 - t)\right\} &\geq \prod_{i=1}^{P_\eta} \mathbb{P}\left\{\overline{B}_i(t) > |c|\,(T_1 - t)\right\} \\
&= \left(\mathbb{P}\left\{\xi > |c|\sqrt{T_1 - t}\right\}\right)^{P_\eta} \geq \left(\mathbb{P}\left\{\xi > |c|\right\}\right)^{P_\eta},
\end{aligned}$$

where ξ is a standard normal random variable. Hence, we can continue our initial inequality as follows

$$\begin{aligned}
\mathbb{P}\left\{\boldsymbol{W}(T_1) > u\mathbf{1}_\eta\right\} &\geq \int_0^{T_1} dF_\mathcal{T}(t)\, \mathbb{P}\left\{\forall \gamma \in \Gamma\colon B_\gamma(T_1) - B_\gamma(t) > c(T_1 - t)\right\} \\
&\geq \left(\mathbb{P}\left\{\xi > |c|\right\}\right)^{P_\eta} \int_0^{T_1} dF_\mathcal{T}(t) \\
&= \left(\mathbb{P}\left\{\xi > |c|\right\}\right)^{P_\eta} \mathbb{P}\left\{\mathcal{T} \leq T_1\right\} \\
&= \left(\mathbb{P}\left\{\xi > |c|\right\}\right)^{P_\eta} \mathbb{P}\left\{\exists\, t \in [0,T_1] : \boldsymbol{W}(t) > u\mathbf{1}_\eta\right\}
\end{aligned}$$

establishing the claim. ☐

The next result shows the solution of the quadratic optimization problem $\Pi_{\Sigma(T)}(\mathbf{1}_\eta)$ (see (2.17)) for our model.

Lemma 5.2. *The unique solution of $\Pi_{\Sigma(T)}(\mathbf{1}_\eta)$ satisfies*

$$\tilde{\boldsymbol{a}} = \mathbf{1}_\eta, \qquad I = \{1, \dots, P_\eta\}, \qquad J = \varnothing.$$

The proof of Lemma 5.2 is given in Appendix.

Let $\delta_u(L) = Lu^{-2}$ for $L > 0$ and denote into two

$$\begin{aligned} M(u,L) &\coloneqq \mathbb{P}\left\{\exists\, t \in [T - \delta_u(L), T] \colon \boldsymbol{W}(t) > u\mathbf{1}_\eta\right\}, \\ m(u,L) &\coloneqq \mathbb{P}\left\{\exists\, t \in [0, T - \delta_u(L)] \colon \boldsymbol{W}(t) > u\mathbf{1}_\eta\right\}. \end{aligned}$$

Clearly,

$$M(u,L) \le \mathbb{P}\left\{\exists\, t \in [0,T] \colon \boldsymbol{W}(t) > u\mathbf{1}_\eta\right\} \le M(u,L) + m(u,L).$$

We will prove that $m(u,L)$ is negligible with respect to $M(u,L)$.

Lemma 5.3. *For fixed $L > 0$, as $u \to \infty$*

$$M(u,L) \sim H(L)\, \mathbb{P}\left\{\boldsymbol{W}(T) > u\mathbf{1}_\eta\right\},$$

where $\boldsymbol{\lambda} = \Sigma^{-1}(T)\mathbf{1}_\eta$, and the constant $H(L)$ is given by

$$H(L) = e^{-\frac{L}{2}\boldsymbol{\lambda}^\top\boldsymbol{\lambda}} \int_{\mathbb{R}^{P_\eta}} \mathbb{P}\left\{\exists\, t \in [0,L] \colon \boldsymbol{\lambda}\boldsymbol{B}^*(t) > \boldsymbol{x}\right\} e^{\sum_{i=1}^{P_\eta} x_i} \mathrm{d}\boldsymbol{x}, \tag{5.15}$$

where $\boldsymbol{B}^*(t) \in \mathbb{R}^{P_\eta}$ is a Brownian motion with independent components.

Remark 5.3. *In view of Lemma 5.2 and Lemma 2.5 (see Appendix) it follows that $\boldsymbol{\lambda} > \mathbf{0}$.*

Proof of Lemma 5.3. Let φ_t denote the pdf of $\boldsymbol{W}(t)$. We can assume that u is large enough to ensure that $i(T - \delta_u(L)) = i(T)$. Then,

$$M(u,L) = u^{-P_\eta} \int_{\mathbb{R}^{P_\eta}} J(u,T,L,\boldsymbol{x})\, \varphi_{T-\delta_u(L)}\left(u\,\mathbf{1}_\eta - \frac{\boldsymbol{x}}{u}\right) d\boldsymbol{x}, \tag{5.16}$$

where

$$J(u,T,L,\boldsymbol{x}) \coloneqq \mathbb{P}\left\{\exists\, t \in [T - \delta_u(L), T] \colon \boldsymbol{W}(t) > u\,\mathbf{1}_\eta \mid \boldsymbol{W}(T - \delta_u(L)) = u\,\mathbf{1}_\eta - \frac{\boldsymbol{x}}{u}\right\}.$$

Next, using Proposition 5.3 we can rewrite the probability $J(u,T,L,\boldsymbol{x})$ as follows

$$\begin{aligned} J(u,T,L,\boldsymbol{x}) &= \mathbb{P}\left\{\exists\, t \in [T - \delta_u(L), T] \colon \boldsymbol{W}(t) - \boldsymbol{W}(T - \delta_u(L)) > \frac{\boldsymbol{x}}{u}\right\} \\ &= \mathbb{P}\left\{\exists\, t \in [T - \delta_u(L), T] \colon \boldsymbol{B}^*(t) - \boldsymbol{B}^*(T - \delta_u(L)) > \frac{\boldsymbol{x}}{u} + \boldsymbol{c}(t - T + \delta_u(L))\right\} \\ &= \mathbb{P}\left\{\exists\, t \in [0, \delta_u(L)] \colon \boldsymbol{B}^*(t) > \frac{\boldsymbol{x}}{u} + \boldsymbol{c}\,t\right\} \\ &= \mathbb{P}\left\{\exists\, t \in [0, L] \colon \boldsymbol{B}^*(t) > \boldsymbol{x} + \frac{\boldsymbol{c}\,t}{u}\right\}, \end{aligned}$$

where $\boldsymbol{B}^*(t) \in \mathbb{R}^{P_\eta}$ is a Brownian motion with independent components. We have

$$\mathbb{P}\left\{\exists\, t \in [0, L] \colon \boldsymbol{B}^*(t) > \boldsymbol{x} + \boldsymbol{c}\, t u^{-1}\right\} \xrightarrow[u\to\infty]{} \mathbb{P}\left\{\exists\, t \in [0, L] \colon \boldsymbol{B}^*(t) > \boldsymbol{x}\right\}$$

for almost all $\boldsymbol{x} \in \mathbb{R}^{P_\eta}$.

Next, let us consider the factor $\varphi_{T-\delta_u(L)}$. We have

$$\varphi_{T-\delta_u(L)}\left(u\,\mathbf{1}_\eta - \frac{\boldsymbol{x}}{u}\right) = (2\pi)^{-P_\eta/2}\left|\Sigma(T-\delta_u(L))\right|^{-1/2} \exp(-G(u, T, L, \boldsymbol{x})),$$

where

$$G(u, T, L, \boldsymbol{x}) := -\frac{1}{2}\left(u\,\mathbf{1}_\eta - \frac{\boldsymbol{x}}{u} + \boldsymbol{c}(T-\delta_u(L))\right)^\top \times \Sigma^{-1}(T-\delta_u(L))\left(u\,\mathbf{1}_\eta - \frac{\boldsymbol{x}}{u} + \boldsymbol{c}(T-\delta_u(L))\right)$$

Using

$$\Sigma^{-1}(T-\delta_u(L)) = \Sigma^{-1}(T) - \delta_u(L)\,\Sigma^{-2}(T) + o\left(\delta_u^2(L)\right),$$

we find that the prefactor converges to $(2\pi)^{-P_\eta/2}\left|\Sigma(T)\right|^{-1/2}$, and we can focus on the asymptotics of the exponent. We have

$$B(u, T, L, \boldsymbol{x}) = \frac{1}{2}\left(u\,\mathbf{1}_\eta - \boldsymbol{c}T\right)^\top \Sigma^{-1}(T)\left(u\,\mathbf{1}_\eta + \boldsymbol{c}\,T\right) + \boldsymbol{x}^\top\Sigma^{-1}(T)\,\mathbf{1}_\eta - \frac{L}{2}\,\mathbf{1}_\eta^\top\Sigma^{-2}(T)\mathbf{1}_\eta + O(L/u),$$

where we used Proposition 5.4. Hence, as $u \to \infty$,

$$\varphi_{T-\delta_u(L)}(u\mathbf{1}_\eta - \boldsymbol{x}/u) \sim \varphi_T(u\mathbf{1}_\eta)\, e^{\boldsymbol{x}^\top\boldsymbol{\lambda}}\, e^{-\frac{L}{2}\boldsymbol{\lambda}^\top\boldsymbol{\lambda}}.$$

Finally, we can bound the pdf under the integral (5.16), for all large enough u, as follows:

$$\frac{\varphi_{T-\delta_u(L)}(u\mathbf{1}_\eta - \boldsymbol{x}/u)}{\varphi_T(u\mathbf{1}_\eta)} \le A\,\exp\left(\boldsymbol{x}^\top\boldsymbol{\lambda}_{\boldsymbol{x}}(\varepsilon)\right) =: \bar{\varphi}(\boldsymbol{x}), \tag{5.17}$$

where $\boldsymbol{\lambda}_{\boldsymbol{x}}(\varepsilon) := \boldsymbol{\lambda} + \operatorname{sign}(\boldsymbol{x})\,\varepsilon > \mathbf{0}$ for all small enough ε and

$$A := \max_{t\in[(\tau_\eta+T)/2, T]}\sqrt{\frac{|\Sigma(T)|}{|\Sigma(t)|}} \times \max_{\substack{t\in[(\tau_\eta+T)/2,T]\\ \boldsymbol{y},\boldsymbol{z}\in[-\mathbf{1},\mathbf{1}]\subset\mathbb{R}^{P_\eta}}} \exp\left(\boldsymbol{y}^\top\Sigma^{-1}(t)(\mathbf{1}_\eta + \boldsymbol{z})\right) < \infty$$

In proving (5.17) we used the fact that both $\boldsymbol{c}Lu^{-1}$ and $\boldsymbol{c}Tu^{-1}$ belong to $\in [-\mathbf{1}, \mathbf{1}]$ for large enough u. The constant A is clearly finite since $\Sigma(t)$ is continuous.

To find the asymptotics of $M(u, L)$, we apply dominated convergence theorem. The probability under the integral $J(u, T, L, \boldsymbol{x})$ can bounded as follows. For $\boldsymbol{x} \in \mathbb{R}^{P_\eta}$, define

$$F_+(\boldsymbol{x}) := \{i \colon x_i > 0\},$$

and two associated sets

$$\mathcal{S}_F := \left\{\boldsymbol{x} \in \mathbb{R}^{P_\eta} \colon F_+(\boldsymbol{x}) = F\right\}, \qquad \mathcal{C} := \left\{\boldsymbol{x} \in \mathbb{R}^d \colon \sum_{i\in F_+(\boldsymbol{x})}(x_i + \varepsilon) < 0\right\}.$$

By Piterbarg inequality (see, e.g., [19, Theorem 8.1]) for each $\boldsymbol{x} \in \mathcal{C}^c$ we have

$$\begin{aligned} J(u,T,L,\boldsymbol{x}) &\le \mathbb{P}\left\{\exists\, t \in [0,L]\ \forall i \in F\colon B_i^*(t) > x_i + \varepsilon\right\} \\ &\le \mathbb{P}\left\{\exists\, t \in [0,L]\colon \sum_{i\in F_+(\boldsymbol{x})} B_i^*(t) > \sum_{i\in F_+(\boldsymbol{x})} (x_i+\varepsilon)\right\} \\ &\le C\left(\sum_{i\in F_+(\boldsymbol{x})}(x_i+\varepsilon)\right)^{\gamma} \exp\left(-\delta \sum_{i\in F_+(\boldsymbol{x})}(x_i+\varepsilon)^2\right) =: \bar{R}(\boldsymbol{x}) \end{aligned}$$

for some positive constants C, γ, δ and ε independent of $\boldsymbol{x}$. Thus, it is enough to show that the integral $\int_{\mathbb{R}^{P_\eta}} \bar{R}(\boldsymbol{x})\,\bar{\varphi}(\boldsymbol{x})dx$ is finite. Setting

$$\mathcal{C}_F := \left\{\boldsymbol{x} \in \mathbb{R}^{P_\eta} \colon \boldsymbol{x} > \boldsymbol{0},\ \sum_{i\in F}(x_i + c) < 0\right\},$$

we obtain

$$\begin{aligned} &\int_{\mathbb{R}^{P_\eta}} \bar{R}(\boldsymbol{x})\,\bar{\varphi}(\boldsymbol{x})\,d\boldsymbol{x} \\ &\quad = CA \sum_{F\subset\{1,\dots,P_\eta\}} \left[\int_{\mathcal{S}_F\setminus C}\left(\sum_{i\in F}(x_i+\varepsilon)\right)^{\gamma} \exp\left(\boldsymbol{x}^\top \boldsymbol{\lambda}_{\boldsymbol{x}}(\varepsilon) - \delta\sum_{i\in F}(x_i+\varepsilon)^2\right) d\boldsymbol{x}\right. \\ &\qquad\qquad \left. + \int_{\mathcal{S}_F\cap\mathcal{C}} \exp\left(\boldsymbol{x}^\top \boldsymbol{\lambda}_{\boldsymbol{x}}(\varepsilon)\right) d\boldsymbol{x}\right] =: CA \sum_{F\subset\{1,\dots,P_\eta\}} \Big[A_1 + A_2\Big]. \end{aligned}$$

The integral A_1 may be estimated as follows:

$$\begin{aligned} A_1 &\le \exp\left(-\frac{1}{4\delta}\,\|\boldsymbol{\lambda}_F + \varepsilon\,\mathbf{1}_F\|_2^2 - \varepsilon\,\mathbf{1}_F^\top(\boldsymbol{\lambda}_F + \varepsilon\,\mathbf{1}_F)\right) \\ &\quad \times \int_{\{\boldsymbol{x}_F>\boldsymbol{0}\}\setminus\mathcal{C}_F} \exp\left(-\delta\left\|\boldsymbol{x}_F + \varepsilon\,\mathbf{1}_F - \frac{1}{2\delta}\,\boldsymbol{\lambda}_{\boldsymbol{x},F}(\varepsilon)\right\|_2^2\right) \|\boldsymbol{x}_F + \varepsilon\,\mathbf{1}_F\|_1^{\gamma}\, d\boldsymbol{x}_F \\ &\quad \times \int_{\boldsymbol{x}_{F^c}\le\boldsymbol{0}_{F^c}} \exp\left(\boldsymbol{x}_{F^c}^\top \boldsymbol{\lambda}_{\boldsymbol{x},F^c}(\varepsilon)\right) d\boldsymbol{x}_{F^c}, \end{aligned}$$

where $\|\cdot\|_p$ denotes the ℓ_p norm. Next, we bound A_2 as follows:

$$A_2 \le \int_{\boldsymbol{x}_F^c\le\boldsymbol{0}_F^c} \exp\left(\boldsymbol{x}_{F^c}^\top\,\boldsymbol{\lambda}_{\boldsymbol{x},F^c}(\varepsilon)\right) d\boldsymbol{x} \times \int_{\mathcal{C}_F} d\boldsymbol{x}_F\,.$$

Combining the two bounds together, we obtain

$$\begin{aligned} \int_{\mathbb{R}^{P_\eta}} \bar{R}(\boldsymbol{x})\,\bar{\varphi}(\boldsymbol{x})\,d\boldsymbol{x} &\le CA \sum_{S\subset\{1,\dots,P_\eta\}} \prod_{i\in F^c}(\lambda_i+\varepsilon)^{-1} \\ &\qquad \times \exp\left(-\frac{1}{4\delta}\,\|\boldsymbol{\lambda}_F + \varepsilon\,\mathbf{1}_F\|_2^2 - \varepsilon\,\mathbf{1}_F^\top(\boldsymbol{\lambda}_F + \varepsilon\,\mathbf{1}_F)\right) A_3(F), \end{aligned}$$

where

$$A_3(F) := \int_{\mathbb{R}^{|F|}} \exp\left(-\delta \left\| \boldsymbol{x}_F + \varepsilon\, \mathbf{1}_F - \frac{1}{2\delta}\, \boldsymbol{\lambda}_{\boldsymbol{x},F}(\varepsilon) \right\|_2^2 \right) \|\boldsymbol{x}_F + \varepsilon\, \mathbf{1}_F\|_1^{\gamma}\, d\boldsymbol{x}_F + \frac{\varepsilon^{|S|}}{|S|!} < \infty$$

for all $\delta > 0$.

Hence, by the dominated convergence theorem, it follows that as $u \to \infty$

$$\begin{aligned} M(u,L) &\sim u^{-P_\eta}\, \varphi_T(u\mathbf{1}_\eta)\, e^{-\frac{L}{2}\boldsymbol{\lambda}^\top\boldsymbol{\lambda}} \int_{\mathbb{R}^{P_\eta}} \mathbb{P}\left\{\exists\, t \in [0,L]:\ \boldsymbol{B}^*(t) > \boldsymbol{x}\right\} e^{-\boldsymbol{x}^\top\boldsymbol{\lambda}} \mathrm{d}\boldsymbol{x} \\ &= \frac{\varphi_T(u\mathbf{1}_\eta)\, e^{-\frac{L}{2}\boldsymbol{\lambda}^\top\boldsymbol{\lambda}}}{\prod_{i=1}^{P_\eta} \lambda_i}\, u^{-P_\eta} \int_{\mathbb{R}^{P_\eta}} \mathbb{P}\left\{\exists\, t \in [0,L]\colon \boldsymbol{\lambda}\boldsymbol{B}^*(t) > \boldsymbol{x}\right\} e^{-\sum_{i=1}^{P_\eta} x_i}\, \mathrm{d}\boldsymbol{x}, \end{aligned} \tag{5.18}$$

which, together with the following asymptotic formula (see [21, Lemma 4.4])

$$\mathbb{P}\left\{\boldsymbol{W}(T) > u\mathbf{1}_\eta\right\} \sim \frac{u^{-P_\eta}}{\prod_{i=1}^{P_\eta} \lambda_i}\, \varphi_T(u\mathbf{1}_\eta), \qquad u \to \infty, \tag{5.19}$$

provides us the claimed assertion.

□

Lemma 5.4. *For $H(L)$ defined in* (5.15) *we have*

$$\mathcal{H}_{P_\eta, 1/\mu_0(T)} = \lim_{L\to\infty} H(L) \in (0,\infty).$$

Proof of Lemma 5.4. Applying the Fubini-Tonelli theorem yields

$$\begin{aligned} H(L) &= e^{-\frac{L}{2}\boldsymbol{\lambda}^\top\boldsymbol{\lambda}} \int_{\mathbb{R}^{P_\eta}} \mathbb{P}\left\{\exists\ t \in [0,L]:\ \boldsymbol{\lambda}\boldsymbol{B}_\eta^*(t) > \boldsymbol{x}\right\} e^{\sum_{i=1}^{P_\eta} x_i} \mathrm{d}\boldsymbol{x} \\ &= \mathbb{E}\left\{ e^{-\boldsymbol{\lambda}^\top \boldsymbol{B}_\eta^*(L) - \frac{L}{2}\boldsymbol{\lambda}^\top\boldsymbol{\lambda} + \boldsymbol{\lambda}^\top \boldsymbol{B}_\eta^*(L)} \int_{\mathbb{R}^{P_\eta}} \mathbb{I}(\exists\, t \in [0,L]\colon \boldsymbol{\lambda}\boldsymbol{B}_\eta^*(t) > \boldsymbol{x}) e^{\sum_{i=1}^{P_\eta} x_i} \mathrm{d}\boldsymbol{x} \right\} \\ &= \mathbb{E}\left\{ e^{-\boldsymbol{\lambda}^\top \boldsymbol{B}_\eta^*(L) - \frac{L}{2}\boldsymbol{\lambda}^\top\boldsymbol{\lambda}} \int_{\mathbb{R}^{P_\eta}} \mathbb{I}(\exists\, t \in [0,L]\colon \boldsymbol{\lambda}\left(\boldsymbol{B}_\eta^*(t) - \boldsymbol{B}_\eta^*(L)\right) > \boldsymbol{x}) e^{\sum_{i=1}^{P_\eta} x_i} \mathrm{d}\boldsymbol{x} \right\} \\ &= \mathbb{E}\left\{ \int_{\mathbb{R}^{P_\eta}} \mathbb{I}(\exists\, t \in [0,L]\colon \boldsymbol{\lambda}\left(\boldsymbol{B}_\eta^*(t) - \boldsymbol{B}_\eta^*(L)\right) - \boldsymbol{\lambda}^2 (t-L) > \boldsymbol{x}) e^{\sum_{i=1}^{P_\eta} \boldsymbol{x}} \mathrm{d}\boldsymbol{x} \right\} \\ &= \int_{\mathbb{R}^{P_\eta}} \mathbb{P}\left\{\exists\, t \in [0,L]\colon \boldsymbol{\lambda}\boldsymbol{B}_\eta^*(t) - \boldsymbol{\lambda}^2 t > \boldsymbol{x}\right\} e^{\sum_{i=1}^{P_\eta} x_i} \mathrm{d}\boldsymbol{x}, \end{aligned}$$

where the second to last step follows from [23, Lem B.6] and the last is a direct consequence of the Brownian motion increments' properties. Hence, $H(L)$ is an increasing function. Using the following inequality, which follows from Proposition 5.7,

$$\begin{aligned} (H(L) - \varepsilon)\mathbb{P}\left\{\boldsymbol{B}_\Gamma(T_1) > u\mathbf{1}_{i(T_1)}\right\} &\le M(u,L) \\ &\le \mathbb{P}\left\{\exists\, t \in [0,T]:\ \boldsymbol{B}_\Gamma(t) > u\mathbf{1}_{i(t)}\right\} \le 2^{P_\eta}\, \mathbb{P}\left\{\boldsymbol{B}_\Gamma(T_1) > u\mathbf{1}_{i(T_1)}\right\} \end{aligned}$$

for any small positive ε and large enough u, we obtain that

$$H(L) \le 2^{P_\eta} + \varepsilon < \infty$$

implies that $H(L)$ is bounded. In order to complete the proof it remains to apply the monotone convergence theorem and notice that according to Proposition 5.5

$$\boldsymbol{\lambda} = \frac{1}{\mu_0(t)} \mathbf{1}_\eta.$$

$\square$

Lemma 5.5. *For $L > 0$ and large enough u,*

$$\frac{m(u, L)}{\mathbb{P}\left\{ \boldsymbol{W}(T) > u\mathbf{1}_\eta \right\}} \leq C_1 e^{-C_2 L}$$

for some positive constants $C_1,\ C_2$, which do not depend on u or L.

Proof of Lemma 5.5. Applying Proposition 5.7

$$\frac{m(u, L)}{\mathbb{P}\left\{ \boldsymbol{W}(T) > u\,\mathbf{1}_\eta \right\}} \leq C \frac{\mathbb{P}\left\{ \boldsymbol{W}(T - \delta_u(L)) > u\,\mathbf{1}_\eta \right\}}{\mathbb{P}\left\{ \boldsymbol{W}(T) > u\,\mathbf{1}_\eta \right\}} = C \frac{\mathbb{P}\left\{ \boldsymbol{W}(T) > \boldsymbol{a}(u, L, T) \right\}}{\mathbb{P}\left\{ \boldsymbol{W}(T) > u\,\mathbf{1}_\eta \right\}},$$

where C defined in Proposition 5.7 and

$$\boldsymbol{a}(u, L, T) := \sqrt{\frac{T}{T - \delta_u(L)}} \Big(u\,\mathbf{1}_\eta + \boldsymbol{c}(T - \delta_u(L)) \Big) - \boldsymbol{c}T.$$

Hence, regarding (5.19), for any $\varepsilon > 0$ and large enough u

$$\begin{aligned} \frac{m(u, L)}{\mathbb{P}\left\{ \boldsymbol{W}(T) > u\,\mathbf{1}_\eta \right\}} &\leq C\,(1 + \varepsilon) \left(\frac{T}{T - \delta_u(L)} \right)^{P_\eta/2} \frac{\varphi_T\left(\boldsymbol{a}(u, L, T) \right)}{\varphi_T\left(u\mathbf{1}_\eta \right)} \\ &\leq C\,(1 + \varepsilon)^{1 + P_\eta/2} \frac{\varphi_T\left(\boldsymbol{a}(u, L, T) \right)}{\varphi_T\left(u\mathbf{1}_\eta \right)}. \end{aligned} \tag{5.20}$$

It remains to bound the quotient $\varphi_T(\boldsymbol{a}(u, L, T))/\varphi_T(u\mathbf{1}_\eta)$. To this end, let us first find the asymptotics of $\boldsymbol{a}(u, L, T)$, as $u \to \infty$

$$\begin{aligned} \boldsymbol{a}(u, L, T) &= u\mathbf{1}_\eta \left(1 - \frac{\delta_u(L)}{T} \right)^{-1/2} - \boldsymbol{c}T \left(1 - \left(1 - \frac{\delta_u(L)}{T} \right)^{1/2} \right) \\ &\sim u\mathbf{1}_\eta \left(1 + \frac{L}{2Tu} \right) + O(Lu^{-2}). \end{aligned}$$

In this computation we used that $\delta_u(L) = Lu^{-2}$. Therefore,

$$\begin{aligned} &\Big(\boldsymbol{a}(u, L, T) + \boldsymbol{c}T \Big)^\top \Sigma^{-1}(T) \Big(\boldsymbol{a}(u, L, T) + \boldsymbol{c}T \Big) \\ &\qquad = \left(u\mathbf{1}_\eta + \boldsymbol{c}T \right)^\top \Sigma^{-1}(T) \left(u\mathbf{1}_\eta + \boldsymbol{c}T \right) + \frac{L}{T} \mathbf{1}_\eta^\top \Sigma^{-1}(T) \mathbf{1}_\eta + O(Lu^{-1}), \end{aligned}$$

which yields

$$\frac{\varphi_T(\boldsymbol{a}(u, L, T))}{\varphi_T(u\mathbf{1}_\eta)} \leq C \, \exp\left(\frac{-L}{2T} \mathbf{1}_\eta^\top \Sigma^{-1} \mathbf{1}_\eta \right).$$

Combining this with (5.20), we obtain the desired result. $\square$

We can now proceed to the proof of Theorem 5.3.

Proof of Theorem 5.3. Using Lemma 5.5, we obtain that for any positive L and large enough u

$$\frac{M(u,L)}{\mathbb{P}\left\{\boldsymbol{W}(T) > u\mathbf{1}_\eta\right\}} \le \frac{\mathbb{P}\left\{\exists\, t \in [0,T] \colon \boldsymbol{W}(t) > u\mathbf{1}_\eta\right\}}{\mathbb{P}\left\{\boldsymbol{W}(T) > u\mathbf{1}_\eta\right\}}$$

$$\le \frac{M(u,L)}{\mathbb{P}\left\{\boldsymbol{W}(T) > u\mathbf{1}_\eta\right\}} + \frac{m(u,L)}{\mathbb{P}\left\{\boldsymbol{W}(T) > u\mathbf{1}_\eta\right\}},$$

where the last term on the right is at most $C_1 \exp(-C_2 L)$ with some positive constants C_1 and C_2. Hence, letting $u \to \infty$ and using Lemma 5.3 we obtain that

$$H(L) \le \liminf_{u\to\infty} \frac{\mathbb{P}\left\{\exists t \in [0,T]:\ \boldsymbol{W}(t) > u\mathbf{1}_\eta\right\}}{\mathbb{P}\left\{\boldsymbol{W}(T) > u\mathbf{1}_\eta\right\}}$$

$$\le \limsup_{u\to\infty} \frac{\mathbb{P}\left\{\exists t \in [0,T]:\ \boldsymbol{W}(t) > u\mathbf{1}_\eta\right\}}{\mathbb{P}\left\{\boldsymbol{W}(T) > u\mathbf{1}_\eta\right\}} \le H(L) + C_1 e^{-C_2 L}.$$

Pushing further $L \to \infty$ and applying Lemma 5.4 we obtain uisng that $C_2 > 0$

$$\mathcal{H}_{P_\eta, 1/\mu_0(T)} \le \liminf_{u\to\infty} \frac{\mathbb{P}\left\{\exists t \in [0,T]:\ \boldsymbol{W}(t) > u\mathbf{1}_\eta\right\}}{\mathbb{P}\left\{\boldsymbol{W}(T) > u\mathbf{1}_\eta\right\}}$$

$$\le \limsup_{u\to\infty} \frac{\mathbb{P}\left\{\exists t \in [0,T]:\ \boldsymbol{W}(t) > u\mathbf{1}_\eta\right\}}{\mathbb{P}\left\{\boldsymbol{W}(T) > u\mathbf{1}_\eta\right\}} \le \mathcal{H}_{P_\eta, 1/\mu_0(T)}.$$

Hence, the claim follows. ☐

Proof of Corollary 5.1. Let us define $N_i^* = \operatorname{essinf}(N_i)$ and take $\tilde{\boldsymbol{N}}_1, \tilde{\boldsymbol{N}}_2 \in \mathbb{N}^\eta$ such that $\tilde{\boldsymbol{N}}_1 \ge \tilde{\boldsymbol{N}}_2$ and $\tilde{\boldsymbol{N}}_1 \ne \tilde{\boldsymbol{N}}_2$. Then we can show the following inequality

$$\mathbb{P}\left\{\exists\, t \in [0,T] \colon \boldsymbol{W}(t) > u\mathbf{1}_\eta \;\middle|\; \boldsymbol{N} = \tilde{\boldsymbol{N}}_1\right\}$$

$$\le \mathbb{P}\left\{\exists\, t \in [0,T] \colon \boldsymbol{W}(t) > u\mathbf{1}_\eta \;\middle|\; \boldsymbol{N} = \tilde{\boldsymbol{N}}_2\right\} \quad (5.21)$$

To show (5.21) it is enough to consider $\tilde{\boldsymbol{N}}_1, \tilde{\boldsymbol{N}}_2$ such that

$$\tilde{\boldsymbol{N}}_1 - \tilde{\boldsymbol{N}}_2 = \boldsymbol{e}_j$$

for some $j \in \{1, \dots, \eta\}$, where $\boldsymbol{e}_j$ is the j-th basis vector in $\mathbb{N}^\eta$. It is easy to see that the process $\boldsymbol{W}(t) \mid \boldsymbol{N} = \tilde{\boldsymbol{N}}_2$ may be obtained from $\boldsymbol{W}(t) \mid \boldsymbol{N} = \tilde{\boldsymbol{N}}_1$ by deleting some branches. Thus, the inequality (5.21) follows.

Hence,

$$\mathbb{P}\left\{\boldsymbol{N} = \boldsymbol{N}^*\right\} \mathbb{P}\left\{\exists\, t \in [0,T] \colon \boldsymbol{W}(t) > u\mathbf{1}_\eta \mid \boldsymbol{N} = \boldsymbol{N}^*\right\}$$

$$\le \mathbb{P}\left\{\exists\, t \in [0,T] \colon \boldsymbol{W}(t) > u\mathbf{1}_\eta\right\}$$

$$\le \mathbb{P}\left\{\boldsymbol{N} = \boldsymbol{N}^*\right\} \mathbb{P}\left\{\exists\, t \in [0,T] \colon \boldsymbol{W}(t) > u\mathbf{1}_\eta \mid \boldsymbol{N} = \boldsymbol{N}^*\right\}$$

$$+ \sum_{i=1}^{\eta} \mathbb{P}\left\{\boldsymbol{N} \ge \boldsymbol{N}^* + \boldsymbol{e}_i\right\} \mathbb{P}\left\{\exists\, t \in [0,T] \colon \boldsymbol{W}(t) > u\mathbf{1}_\eta \mid \boldsymbol{N} = \boldsymbol{N}^* + \boldsymbol{e}_i\right\},$$

and the claim follows from Theorem 5.3 and the fact that

$$\mathbb{P}\left\{\boldsymbol{W}(T) > u\mathbf{1}_\eta \mid \boldsymbol{N} = \boldsymbol{N}^* + \boldsymbol{e}_i\right\} = o\left(\mathbb{P}\left\{\boldsymbol{W}(T) > u\mathbf{1}_\eta \mid \boldsymbol{N} = \boldsymbol{N}^*\right\}\right).$$

To show the latter, we use Proposition 5.6 as follows. Combining

$$(P_\eta \mid \boldsymbol{N} = \boldsymbol{N}^* + \boldsymbol{e}_i) = \left(1 + \frac{1}{N_i^*}\right)(P_\eta \mid \boldsymbol{N} = \boldsymbol{N}^*),$$

with

$$\begin{aligned}(\mu_0(T) \mid \boldsymbol{N} = \boldsymbol{N}^* + \boldsymbol{e}_i) &= (\mu_0(T) \mid \boldsymbol{N} = \boldsymbol{N}^*) + \sum_{l=1}^{i}(\tau_l - \tau_{l-1})\prod_{\substack{j=l\\ j\neq i}}^{\eta} N_j \\ &= (\mu_0(T) \mid \boldsymbol{N} = \boldsymbol{N}^*) + \frac{1}{N_i}\sum_{l=1}^{i}(\tau_l - \tau_{l-1})\prod_{j=l}^{\eta} N_j \\ &\leq \left(1 + \frac{1}{N_i}\right)(\mu_0(T) \mid \boldsymbol{N} = \boldsymbol{N}^*),\end{aligned}$$

we obtain

$$\left(\frac{P_\eta}{\mu_0(T)} \,\middle|\, \boldsymbol{N} = \boldsymbol{N}^* + \boldsymbol{e}_i\right) \leq \left(\frac{P_\eta}{\mu_0(T)} \,\middle|\, \boldsymbol{N} = \boldsymbol{N}^*\right),$$

and invoke Proposition 5.6. □

Proof of Corollary 5.2. Using Theorem 5.3, we obtain

$$\begin{aligned}\mathbb{P}\left\{u^2(T - \mathcal{T}(u)) \geq x \,\middle|\, \mathcal{T}(u) \leq T - \frac{y}{u^2}\right\} &= \frac{\mathbb{P}\left\{\mathcal{T}(u) \leq T - xu^{-2}\right\}}{\mathbb{P}\left\{\mathcal{T}(u) \leq T - yu^{-2}\right\}} \\ &\sim \frac{\mathcal{H}_{P_\eta, 1/\mu_0(T - yu^{-2})}\,\mathbb{P}\left\{\boldsymbol{W}(T - xu^{-2}) > u\,\mathbf{1}_\eta\right\}}{\mathcal{H}_{P_\eta, 1/\mu_0(T - xu^{-2})}\,\mathbb{P}\left\{\boldsymbol{W}(T - yu^{-2}) > u\,\mathbf{1}_\eta\right\}} \sim \frac{\mathbb{P}\left\{\boldsymbol{W}(T - xu^{-2}) > u\,\mathbf{1}_\eta\right\}}{\mathbb{P}\left\{\boldsymbol{W}(T - yu^{-2}) > u\,\mathbf{1}_\eta\right\}}.\end{aligned}$$

Applying Proposition 5.6, we find that the latter is asymptotically equivalent to

$$\begin{aligned}&\sim \frac{\mu_0^{P_\eta - 1/2}(T - yu^{-2})\prod_{v=1}^{\eta}\mu_v^{(P_v - P_{v-1})/2}(T - xu^{-2})}{\mu_0^{P_\eta - 1/2}(T - yu^{-2})\prod_{v=1}^{\eta}\mu_v^{(P_v - P_{v-1})/2}(T - yu^{-2})} \\ &\qquad\qquad \times \exp\left(-\frac{(u + c(T - xu^{-2}))^2 P_\eta}{2\mu_0(T - xu^{-2})} + \frac{(u + c(T - yu^{-2}))^2 P_\eta}{2\mu_0(T - yu^{-2})}\right).\end{aligned}$$

Since the pre-exponential factor clearly tends to 1, it remains to compute the asymptotics of the exponent. We have

$$\begin{aligned}&-\frac{(u + c(T - xu^{-2}))^2 P_\eta}{2\mu_0(T - xu^{-2})} + \frac{(u + c(T - yu^{-2}))^2 P_\eta}{2\mu_0(T - yu^{-2})} \\ &\qquad\qquad \sim -\frac{(u + cT)^2 P_\eta}{2\mu_0(T - xu^{-2})\mu_0(T - yu^{-2})}\Big[\mu_0(T - yu^{-2}) - \mu_0(T - xu^{-2})\Big].\end{aligned}$$

It thus remains to note that

$$\mu_0(T - yu^{-2}) - \mu_0(T - xu^{-2}) \sim u^{-2}(x - y) \quad \text{and} \quad \mu_0(T - xu^{-2}),\ \mu_0(T - yn^{-2}) \sim \mu_0(T)$$

to conclude the proof. □

5.4.5 Proof of Theorem 5.4

We note that

$$\begin{aligned}\mathbb{P}\left\{\exists\, i \in \{1,\dots,M\},\, t\in[0,T] \colon \boldsymbol{W}_i(t) > u\boldsymbol{1}_{\eta_i}\right\} \\ \geq \sum_{i=1}^{M} \mathbb{P}\left\{\exists\, t\in[0,T] \colon \boldsymbol{W}_i(t) > u\boldsymbol{1}_{\eta_i}\right\} - \sum_{\substack{i,j=1\\ i\neq j}}^{M} \mathbb{P}\left\{\begin{matrix}\exists\, t_1\in[0,T] \colon \boldsymbol{W}_i(t_1) > u\boldsymbol{1}_{\eta_i} \\ \exists\, t_2\in[0,T] \colon \boldsymbol{W}_j(t_2) > u\boldsymbol{1}_{\eta_j}\end{matrix}\right\}\end{aligned}$$

and

$$\mathbb{P}\left\{\exists\, i \in \{1,\dots,M\},\, t\in[0,T] \colon \boldsymbol{W}_i(t) > u\boldsymbol{1}_{\eta_i}\right\} \leq \sum_{i=1}^{M} \mathbb{P}\left\{\exists\, t\in[0,T] \colon \boldsymbol{W}_i(t) > u\boldsymbol{1}_{\eta_i}\right\}.$$

Using that $\boldsymbol{W}_i$ are independent,

$$\begin{aligned}&\mathbb{P}\left\{\exists\, t_1,\, t_2\in[0,T] \colon \boldsymbol{W}_i(t_1) > u\boldsymbol{1}_{\eta_i},\ \boldsymbol{W}_j(t_2) > u\boldsymbol{1}_{\eta_j}\right\} \\ &\quad= \mathbb{P}\left\{\exists\, t\in[0,T] \colon \boldsymbol{W}_i(t) > u\boldsymbol{1}_{\eta_i}\right\} \mathbb{P}\left\{\exists\, t\in[0,T] \colon \boldsymbol{W}_j(t) > u\boldsymbol{1}_{\eta_j}\right\} \\ &\qquad+ o\left(\mathbb{P}\left\{\exists\, t\in[0,T] \colon \boldsymbol{W}_i(t) > u\boldsymbol{1}_{\eta_i}\right\} + \mathbb{P}\left\{\exists\, t\in[0,T] \colon \boldsymbol{W}_j(t) > u\boldsymbol{1}_{\eta_j}\right\}\right).\end{aligned}$$

We find

$$\mathbb{P}\left\{\exists\, i \in \{1,\dots,M\},\, t\in[0,T] \colon \boldsymbol{W}_i(t) > u\boldsymbol{1}_{\eta_i}\right\} \sim \sum_{i=1}^{M} \mathbb{P}\left\{\exists\, t\in[0,T] \colon \boldsymbol{W}_i(t) > u\boldsymbol{1}_{\eta_i}\right\}.$$

Finally, Lemma 5.1 implies that

$$\sum_{i\notin A} \mathbb{P}\left\{\exists\, t\in[0,T] \colon \boldsymbol{W}_i(t) > u\boldsymbol{1}_{\eta_i}\right\} = o\left(\sum_{i\in A} \mathbb{P}\left\{\exists\, t\in[0,T] \colon \boldsymbol{W}_i(t) > u\boldsymbol{1}_{\eta_i}\right\}\right),$$

establishing the proof.

5.5 Appendix

Proof of Proposition 5.2. Using mathematical induction, we shall show that for all $i \in \{1,\dots,N\}$ holds

$$B_\gamma(\tau_i) = B_0^*(\tau_1) + \sum_{j=1}^{i-1} (\boldsymbol{B}_j^*)_{(\gamma \bmod P_j)+1}(\tau_{j+1}-\tau_j). \tag{5.22}$$

Since for $i=1$ the claim is clear, assuming it holds for $i=k$, we have for $i=k+1$

$$\begin{aligned}B_\gamma(\tau_{k+1}) &= \left(\widetilde{\boldsymbol{B}}_\Gamma\right)_{(\gamma \bmod P_k)+1}(\tau_{k+1}) &(5.23)\\ &= \begin{pmatrix}\widetilde{\boldsymbol{B}}_\Gamma(\tau_k)\\ \vdots \\ \widetilde{\boldsymbol{B}}_\Gamma(\tau_k)\end{pmatrix}_{(\gamma \bmod P_k)+1} + \left(\boldsymbol{B}_k^*\right)_{(\gamma \bmod P_k)+1}(\tau_{k+1}-\tau_k). &(5.24)\end{aligned}$$

For the first term we have

$$\begin{pmatrix} \widetilde{\boldsymbol{B}}_\Gamma(\tau_k) \\ \vdots \\ \widetilde{\boldsymbol{B}}_\Gamma(\tau_k) \end{pmatrix}_{(\gamma \bmod P_k)+1} = \big(\widetilde{\boldsymbol{B}}_\Gamma(\tau_k)\big)_{\big((\gamma_k \bmod P_k) \bmod P_{k-1}\big)+1}$$

$$= \big(\widetilde{\boldsymbol{B}}_\Gamma(\tau_k)\big)_{(\gamma_k \bmod P_{k-1})+1} = B_\gamma(\tau_k).$$

By the induction hypothesis,

$$B_\gamma(\tau_k) = B_0^*(\tau_1) + \sum_{j=1}^{k-1} \big(\boldsymbol{B}_j^*\big)_{(\gamma \bmod P_j)+1}(\tau_{j+1} - \tau_j). \tag{5.25}$$

Subsume the second term of (5.23) into the sum (5.25) and note that this conclues the proof of this assertion for $t = \tau_{k+1}$. Similarly, if $t \in (\tau_1, T]$, then

$$B_\gamma(t) = \begin{pmatrix} \widetilde{\boldsymbol{B}}_\Gamma(\tau_{i(t)}) \\ \vdots \\ \widetilde{\boldsymbol{B}}_\Gamma(\tau_{i(t)}) \end{pmatrix}_{(\gamma \bmod P_{i(t)})+1} + \big(\boldsymbol{B}_k^*\big)_{(\gamma \bmod P_{i(t)})+1}(t - \tau_{i(t)}),$$

and therefore

$$B_\gamma(t) = B_0^*(\tau_1) + \sum_{j=1}^{i(t)-1} \big(\boldsymbol{B}_j^*\big)_{(\gamma \bmod P_j)+1}(\tau_{j+1} - \tau_j) + \big(\boldsymbol{B}_{i(t)}^*\big)_{(\gamma \bmod P_{i(t)})+1}(t - \tau_{i(t)}).$$

Thus, the claim (5.5) follows, justifying that B_γ is a Brownian motion.

Consider now the assertion (5.6). Let $t_1 \le \tau_{\kappa(\gamma_1,\gamma_2)}$. Using that $B_{\gamma_1}(t) = B_{\gamma_2}(t)$ for any $t \le \tau_{\kappa(\gamma_1,\gamma_2)}$, we have

$$B_{\gamma_1}(t_1) = B_{\gamma_2}(t_1).$$

Since B_{γ_2} is a Brownian motion, we obtain

$$\mathrm{Cov}(B_{\gamma_1}(t_1), B_{\gamma_2}(t_2)) = \mathrm{Cov}(B_{\gamma_2}(t_1), B_{\gamma_2}(t_2)) = \min\{t_1, t_2\} = \min\{t_1, t_2, \tau_{\kappa(\gamma_1,\gamma_2)}\}.$$

The same holds in the case $t_2 \le \tau_{\kappa(\gamma_1,\gamma_2)}$. In the case $\tau_{\kappa(\gamma_1,\gamma_2)} < t_1, t_2$, using (5.5), we can see that

$$B_{\gamma_1}(t_1) = B_0^*(\tau_1) + \sum_{j=1}^{i(t_1)-1} \big(\boldsymbol{B}_j^*\big)_{(\gamma_1 \bmod P_j)+1}(\tau_{j+1} - \tau_j)$$
$$+ \big(\boldsymbol{B}_{i(t_1)}^*\big)_{(\gamma_1 \bmod P_{i(t_1)})+1}(t_1 - \tau_{i(t_1)}).$$

Next, we split the middle sum at the branches' separation point $\kappa(\gamma_1, \gamma_2)$ and use the fact that

$$B_0^*(\tau_1) + \sum_{j=1}^{\kappa(\gamma_1,\gamma_2)-1} \big(\boldsymbol{B}_j^*\big)_{(\gamma_1 \bmod P_j)+1}(\tau_{j+1} - \tau_j) = B_{\gamma_1}(\tau_{\kappa(\gamma_1,\gamma_2)})$$

to obtain

$$B_{\gamma_1}(t_1)B_{\gamma_1}(\tau_{\kappa(\gamma_1,\gamma_2)}) + \sum_{j=\kappa(\gamma_1,\gamma_2)}^{i(t_1)-1} \big(\boldsymbol{B}_j^*\big)_{(\gamma_1 \bmod P_j)+1}(\tau_{j+1}-\tau_j) \\ + \big(\boldsymbol{B}_{i(t_1)}^*\big)_{(\gamma_1 \bmod P_{i(t_1)})+1}(t_1-\tau_{i(t_1)}).$$

By the same reason,

$$B_{\gamma_2}(t_2) = B_{\gamma_2}(\tau_{\kappa(\gamma_1,\gamma_2)}) + \sum_{j=\kappa(\gamma_1,\gamma_2)}^{i(t_2)-1} \big(\boldsymbol{B}_j^*\big)_{(\gamma_2 \bmod P_j)+1}(\tau_{j+1}-\tau_j) \\ + \big(\boldsymbol{B}_{i(t_2)}^*\big)_{(\gamma_2 \bmod P_{i(t_2)})+1}(t_2-\tau_{i(t_2)}).$$

We have used the fact that for all indices $j \geq \kappa(\gamma_1,\gamma_2)$ holds $(\gamma_1 \bmod P_j) \neq (\gamma_2 \bmod P_j)$, so the sums

$$\sum_{j=\kappa(\gamma_1,\gamma_2)}^{i(t_1)-1} \big(\boldsymbol{B}_j^*\big)_{(\gamma_1 \bmod P_j)+1}(\tau_{j+1}-\tau_j) + \big(\boldsymbol{B}_{i(t_1)}^*\big)_{(\gamma_1 \bmod P_{i(t_1)})+1}(t_1-\tau_{i(t_1)})$$

and

$$\sum_{j=\kappa(\gamma_1,\gamma_2)}^{i(t_2)-1} \big(\boldsymbol{B}_j^*\big)_{(\gamma_2 \bmod P_j)+1}(\tau_{j+1}-\tau_j) + \big(\boldsymbol{B}_{i(t_2)}^*\big)_{(\gamma_2 \bmod P_{i(t_2)})+1}(t_2-\tau_{i(t_2)})$$

are independent. Additionally, each of them is independent of $B_{\gamma_1}(\tau_{\kappa(\gamma_1,\gamma_2)})$, which is equal to $B_{\gamma_2}(\tau_{\kappa(\gamma_1,\gamma_2)})$. Hence,

$$\mathrm{Cov}(B_{\gamma_1}(t_1), B_{\gamma_2}(t_2)) = \mathrm{Cov}(B_{\gamma_1}(\tau_{\kappa(\gamma_1,\gamma_2)}), B_{\gamma_2}(\tau_{\kappa(\gamma_1,\gamma_2)})) \\ = \mathrm{Var}(B_{\gamma_1}(\tau_{\kappa(\gamma_1,\gamma_2)})) = \tau_{\kappa(\gamma_1,\gamma_2)}.$$

This establishes (5.6). □

Proof of Proposition 5.5. We are going to prove the claim of the theorem by induction in $t \in \{\tau_0, \ldots, \tau_\eta\}$. The case $t = \tau_0$ is clear. Assume that the claim is true for $t = \tau_k$. We claim that the eigenvalues of

$$\begin{pmatrix} \Sigma(\tau_k) & \ldots & \Sigma(\tau_k) \\ \vdots & & \vdots \\ \Sigma(\tau_k) & \ldots & \Sigma(\tau_k) \end{pmatrix}$$

are the eigenvalues of $\Sigma(\tau_k)$ multiplied by N_k. Moreover, they are of at least the same multiplicity. Indeed, for any eigenvalue μ and a corresponding eigenvector $\boldsymbol{v} \in \mathbb{R}^{P_{k-1}}$ of $\Sigma(\tau_k)$ we can construct a vector $\boldsymbol{v}^* = (\boldsymbol{v}^\top, \ldots, \boldsymbol{v}^\top)^\top \in \mathbb{R}^{P_k}$, which satisfies

$$\begin{pmatrix} \Sigma(\tau_k) & \ldots & \Sigma(\tau_k) \\ \vdots & & \vdots \\ \Sigma(\tau_k) & \ldots & \Sigma(\tau_k) \end{pmatrix} \begin{pmatrix} \boldsymbol{v} \\ \vdots \\ \boldsymbol{v} \end{pmatrix} = \begin{pmatrix} N_k\Sigma(\tau_k)\boldsymbol{v} \\ \vdots \\ N_k\Sigma(\tau_k)\boldsymbol{v} \end{pmatrix} = \begin{pmatrix} N_k\mu\boldsymbol{v} \\ \vdots \\ N_k\mu\boldsymbol{v} \end{pmatrix} = N_k\,\mu\,\boldsymbol{v}^*.$$

In particular, it means that as $\mathbf{1}_{k-1}$ is an eigenvector of $\Sigma(\tau_k)$ with the eigenvalue $\mu_0(\tau_k)$, then $\mathbf{1}_k$ is an eigenvector of the matrix mentioned above with the corresponding eigenvalue $N_k\,\mu_0(\tau_k)$. Since

$$\mathrm{rank}\begin{pmatrix} \Sigma(\tau_k) & \ldots & \Sigma(\tau_k) \\ \vdots & & \vdots \\ \Sigma(\tau_k) & \ldots & \Sigma(\tau_k) \end{pmatrix} = \mathrm{rank}\,\Sigma(\tau_k) = P_{k-1},$$

0 is an eigenvalue of the matrix mentioned above of multiplicity $P_k - P_{k-1}$. Using Proposition 5.4 for any eigenvalue $\mu_v(\tau_k)$ of $\Sigma(\tau_k)$ we can find an eigenvalue $\mu_v(\tau_{k+1})$ of $\Sigma(\tau_{k+1})$ as follows

$$\begin{aligned}
\mu_v(\tau_{k+1}) &= N_k \mu_v(\tau_k) + (\tau_{k+1} - \tau_k) \\
&= (\tau_{k+1} - \tau_k) + N_k \left((\tau_k - \tau_{k-1}) + \sum_{l=v+1}^{k-1} (\tau_l - \tau_{l-1}) \prod_{j=l}^{k-1} N_j \right) \\
&= (\tau_{k+1} - \tau_k) + N_k(\tau_k - \tau_{k-1}) + \sum_{l=v+1}^{k-1} (\tau_l - \tau_{l-1}) \prod_{j=l}^{k} N_j \\
&= (\tau_{k+1} - \tau_k) + \sum_{l=v+1}^{k} (\tau_l - \tau_{l-1}) \prod_{j=l}^{k} N_j.
\end{aligned}$$

The multiplicity of $\mu_v(\tau_{k+1})$ equals to the multiplicity of $\mu_v(\tau_k)$, i.e. $P_v - P_{v-1}$. In particular, for $i = 0$ we have that $\mu_0(\tau_{k+1})$ has exactly one eigenvector $\mathbf{1}_k$. Additionally, $\Sigma(\tau_{k+1})$ has the eigenvalue $\tau_{k+1} - \tau_k$ with multiplicity $P_k - P_{k-1}$. Hence, the claim follows for the matrix $\Sigma(\tau_{k+1})$. Finally, using Proposition 5.4 we can find eigenvalues and eigenvectors of $\Sigma(t)$ if we know them for $\Sigma(\tau_{i(t)})$. □

Proof of Proposition 5.6. Using that $\boldsymbol{\lambda} = \Sigma^{-1}\mathbf{1}_\eta$ and Proposition 5.5 we obtain

$$\boldsymbol{\lambda} = \frac{1}{\mu_0(T)} \mathbf{1}_\eta, \qquad \mathbf{1}_\eta^\top \Sigma^{-1}(T) \mathbf{1}_\eta = \frac{P_\eta}{\mu_0(T)}.$$

Additionally, using Proposition 5.5 we know that

$$|\Sigma(T)| = \mu_0(T) \prod_{v=1}^{\eta} \mu_v^{P_v - P_{v-1}}(T).$$

Hence, the claim follows from (5.19):

$$\begin{aligned}
\mathbb{P}\left\{ \boldsymbol{B}_\Gamma(T) - cT\mathbf{1}_\eta > u\mathbf{1}_\eta \right\} &\sim \frac{1}{\prod_{i=1}^{P_\eta} \lambda_i} u^{-P_\eta} \varphi_T(u\mathbf{1}_\eta) \\
&= \frac{u^{-P_\eta}}{\prod_{i=1}^{P_\eta} \lambda_i} \frac{1}{(2\pi)^{P_\eta/2} |\Sigma(T)|^{1/2}} \exp\left(-\frac{1}{2}(u + cT)^2 \mathbf{1}_\eta^\top \Sigma^{-1}(T) \mathbf{1}_\eta \right).
\end{aligned}$$

Recall that $\varphi(T)$ stands for the pdf of $\boldsymbol{B}_\Gamma(T) - cT\mathbf{1}_\eta$. □

Proof of Lemma 5.2. Note that any element $\gamma \in \Gamma$ has unique representation of the following form

$$\gamma = \sum_{i=1}^{\eta} a_i P_{i-1},$$

where $a_i \in \{0, 1, \dots, N_i - 1\}$. Hence, the map

$$\gamma \mapsto (a_1(\gamma), a_2(\gamma), \dots, a_\eta(\gamma))$$

is a bijection. Let us introduce the following class of permutations on Γ:

$$\pi_{j,b,c}(\gamma) = \gamma', \qquad i \in \{1, 2, \dots, \eta\}, \quad b, c \in \{0, 1, \dots, N_i - 1\},$$

where for any $i \in \{1, 2, \ldots, \eta\}$

$$a_i(\gamma') = \begin{cases} a_i(\gamma), & \text{if } i \neq j, \\ a_j(\gamma), & \text{if } i = j, \ a_j(\gamma) \neq c \text{ and } a_j(\gamma) \neq b, \\ b, & \text{if } i = j \text{ and } a_j(\gamma) = c, \\ c, & \text{if } i = j \text{ and } a_j(\gamma) = b. \end{cases}$$

Next, define a new Gaussian random vector $\tilde{\boldsymbol{X}}(j, b, c) = (\tilde{X}_0, \ldots, \tilde{X}_{P_\eta - 1}) \in \mathbb{R}^{P_\eta}$

$$\tilde{X}_i = B_{\pi_{j,b,c}(i)}(T)$$

and denote its covariance function by $\Sigma(j, b, c)$. Consider a new quadratic programming problem (see Lemma 2.5)

$$\Pi_{\Sigma(j,b,c)}(\mathbf{1}_\eta).$$

Define its corresponding sets by $I(j, b, c)$ and $J(j, b, c)$. Using that our vector $\tilde{\boldsymbol{X}}(j, b, c)$ is a permutation of $\boldsymbol{B}_\Gamma(T)$,

$$I(j, b, c) = \pi_{j,b,c}(I).$$

On the other hand, from the definition of $\pi_{j,b,c}$ we have that for any $i \in \{1, \ldots, \eta\}$

$$a_i(\gamma_1) = a_i(\gamma_2) \iff a_i(\pi_{j,b,c}(\gamma_1)) = a_i(\pi_{j,b,c}(\gamma_2)).$$

Using the formula for γ in terms of $a_i(\gamma)$ we obtain that

$$\gamma_1 = \gamma_2 \bmod P_i \iff \forall\, l \in \{1, 2, \ldots, i\} \ a_l(\gamma_1) = a_l(\gamma_2).$$

Combining these two facts, we find

$$\gamma_1 = \gamma_2 \bmod P_i \iff \pi_{j,b,c}(\gamma_1) = \pi_{j,b,c}(\gamma_2) \bmod P_i,$$

which implies that

$$\kappa(\gamma_1, \gamma_2) = \kappa(\pi_{j,b,c}(\gamma_1), \pi_{j,b,c}(\gamma_2)).$$

Consequently, we have

$$\begin{aligned} \operatorname{Cov}(\tilde{X}_{\gamma_1}, \tilde{X}_{\gamma_2}) &= \operatorname{Cov}(B_{\pi_{j,b,c}(\gamma_1)}(T), B_{\pi_{j,b,c}(\gamma_2)}(T)) \\ &= \min\{T, \tau_{\kappa(\pi_{j,b,c}(\gamma_1), \pi_{j,b,c}(\gamma_2))}\} = \min\{T, \tau_{\kappa(\gamma_1, \gamma_2)}\} = \operatorname{Cov}(B_{\gamma_1}(T), B_{\gamma_2}(T)). \end{aligned}$$

Hence $\Sigma(j, b, c) = \Sigma(T)$, and

$$I(j, b, c) = I.$$

It means that for any $j \in \{1, \ldots, \eta\}$, $b, c \in \{0, \ldots, N_j\}$

$$\pi_{j,b,c}(I) = I.$$

Finally, let us notice that for any γ_1, $\gamma_2 \in \Gamma$ we can find a sequence of the permutations of the class mentioned above which send γ_1 into γ_2:

$$\gamma_2 = \pi_{1,a_1(\gamma_1),a_1(\gamma_2)}(\pi_{2,a_2(\gamma_1),a_2(\gamma_2)}(\cdots \pi_{\eta,a_\eta(\gamma_1),a_\eta(\gamma_2)}(\gamma_1)\cdots)).$$

This implies that set I can either be empty, or contain all elements of Γ. Since by Lemma 2.5 it cannot be empty, the claim follows. $\square$

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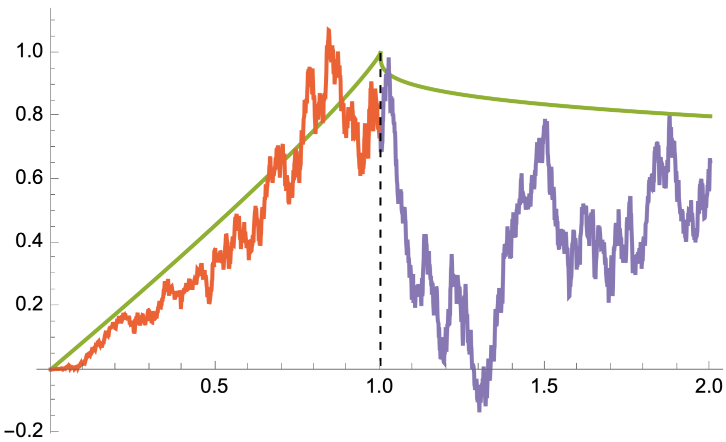
Chapter 6

Parisian ruin with power-asymmetric variance near the optimal point with application to many-inputs proportional reinsurance



This chapter investigates the Parisian ruin probability for a class of Gaussian processes with power-asymmetric behavior of the variance near the unique optimal point. We derive the exact asymptotics as the initial capital tends to infinity and extend the previous result [1] to the case when the length of Parisian interval is of Pickands scale. As a primary application, we extend the recent result [2] on the many inputs proportional reinsurance fractional Brownian motion risk model to the Parisian ruin.

Pavel Ievlev, *Parisian ruin with power-asymmetric variance near the optimal point with application to many-inputs proportional reinsurance*, Stochastic Models **40** (2024), no. 3, pp. 518–535. MR4777226

6.1 Introduction

Consider the following reinsurance scheme: d companies share premiums and one claim process proportionally. Suppose that the risk process $\boldsymbol{R}(t)$ is composed of a large number of i.i.d. sub-risk processes $\boldsymbol{R}^{(i)}(t)$ representing independent businesses, and let each $\boldsymbol{R}^{(i)}$ be driven by a fractional Brownian motion $B_H(t)$. That is, let

$$\boldsymbol{R}_N(t) = \sum_{i=1}^{N} \boldsymbol{R}^{(i)}(t), \quad \text{where} \quad \boldsymbol{R}^{(i)}(t) = \boldsymbol{m} + \boldsymbol{\mu} t - \boldsymbol{\sigma} B_H^{(i)}(t),$$

where $B_H^{(i)}, i \geq 1$ are independent fractional Brownian motions and $\boldsymbol{m}, \boldsymbol{\mu}, \boldsymbol{\sigma} \in \mathbb{R}^d$. In a recent contribution [2], the authors derived the exact asymptotics of the simultaneous ruin probability

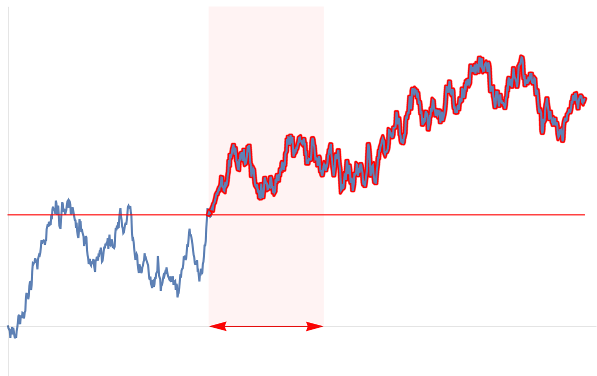
$$\mathbb{P}\left\{\exists t \in [0,T] \colon \boldsymbol{R}_N(t) < 0\right\}, \quad N \to \infty$$

in the case of $d = 2$. In the present work we shall concentrate on the simultaneous Parisian ruin probability

$$\widetilde{\Pi}(N) = \mathbb{P}\left\{\mathcal{P}_{[0,S],[0,T_N]}(\boldsymbol{R}_N) < 0\right\}, \quad \text{where} \quad \mathcal{P}_{E,F}(\boldsymbol{Z}) = \max_{i=1,\dots,d} \sup_{t \in E} \inf_{s \in F} Z_i(t+s) \tag{6.1}$$

for arbitrary d.

Parisian stopping times have been first introduced in relation to barrier options in mathematical finance, see [3], and since then attracted substantial interest. For the applications to actuarial risk theory, we refer to [4], where risk process is treated as a surplus process of an insurance company with initial capital u.



Parisian ruin is recognized if the process has spent a sufficient amount of time above the threshold.

In opposition to the well-studied classical ruin, when the failure is recognized at the moment of surplus hitting zero, the Parisian ruin is recognized only if the surplus process has spent a sufficient, pre-specified amount of time below zero. We refer to [5, 6, 7, 8, 9] and the references therein for analysis of Parisian ruin in the one-dimensional Lévy surplus model.

In the univariate Gaussian setup, Parisian ruin has been investigated in [10] for self-similar Gaussian processes and in [1] for general Gaussian processes, satisfying some standard assumptions (see [11]). Another interesting univariate case is of Parisian ruin over *discrete sets*. In [12], the authors have proved that for the Brownian motion and *equidistant grid* the asymptotics differs from the continuous one by some constant factor.

There are many possible extensions of the notion of the Parisian ruin to multivariate risk processes, such as *simultaneous Parisian ruin*, when all the components of a multivariate process should plunge below zero *at the same time* and remain there long enough for the ruin to be attested. This problem has recently been studied in [13] for the case when the risk process consists of two correlated Brownian motions. Another possible extension is the *joint* or *non-simultaneous Parisian ruin*, when ruin is attested if all the components experience Parisian ruin during some interval of time, but *not necessarily at the same time.* This problem has been studied in [14], also for the bivariate Brownian motion with $\rho \in (-1, 1)$. A third possible extension would be the notion of "at least one" ruin, suggested in the classical ruin context in [2]: the ruin is declared if *either* of the processes has a Parisian ruin over time.

In this paper, we derive the exact asymptotics as $u \to \infty$ of the one-dimensional[1] Parisian ruin probability

$$\Pi(u) = \mathbb{P}\left\{\mathcal{P}_{[-S,S],[0,T_u]}(Z) > u\right\}$$

for $T_u \to 0$ at some specified rate and a class of Gaussian processes with correlation structure

$$\mathrm{Corr}(Z(t), Z(s)) = 1 - D|t-s|^{\alpha} + o\left(|t-s|^{\alpha}\right)$$

and a unique optimal point $t_* = 0$ of the variance with asymmetric behaviour near this point:

$$\sigma(t) = 1 - A_{\pm}|t|^{\gamma_{\pm}}\Big(1 + o(1)\Big) \quad \text{as } t \to \pm 0,$$

from which we further derive the exact asymptotics of the many-inputs Parisian ruin probability (6.1).

The asymptotic behaviour of $\Pi(u)$ for such class of processes is of interest by itself. Similar problems have recently been studied in [10] and [1]. Our findings account for the previously discarded type of the Talagrand case with $T > 0$ (see Section 6.2) and discover the new type of asymptotics therein:

$$\Pi(u) = e^{-\min\{A_- T^{\gamma_-}, A_+ T^{\gamma_+}\}}\Psi(u),$$

which rather surprisingly happens only if $\gamma_+ < \gamma_-, \alpha$ and does not happen if $\gamma_- < \gamma_+, \alpha$. Here Ψ denotes the survival function of a standard normal random variable $N(0,1)$.

The paper is organized as follows. In Section 6.2 we present our main findings. Theorem 6.1 provides the exact asymptotics of $\Pi(u)$ for the general Gaussian process with power-asymmetric behaviour of the variance near the optimal point and under some assumption on the speed $T_u \to 0$ convergence. It covers the previously unaccounted for case when the size of Parisian interval is equivalent to the Pickands scale (see Section 6.3.1) of the process. Corollary 6.1 contains the exact asymptotics of the many-inputs Parisian ruin probability. The proof of Theorem 6.1 is presented in a separate Section 6.3. All known results and technical details are relegated to the Appendix.

6.2 Main results

In this section, we first explain how to rewrite many-inputs ruin probability in a form suitable for applying Theorem 6.1, then specify the assumptions under which the general theorem works and conclude with deriving the exact asymptotics of the many-inputs proportional reinsurance ruin probability (6.1).

Observe that by properties of Gaussian distribution

$$\boldsymbol{R}_N \stackrel{d}{=} \boldsymbol{m}N + \boldsymbol{\mu}Nt - \boldsymbol{\sigma}\sqrt{N}B_H(t),$$

[1] By *one-dimensional* we mean that the process Z takes values in $\mathbb{R}^d$ with $d = 1$.

we can rewrite the ruin probability (6.1) as

$$\widetilde{\Pi}_{[0,S]}(N) = \mathbb{P}\left\{\mathcal{P}_{[0,S],[0,T_N]}(Z) > \sqrt{N}\right\}, \quad \text{where} \quad Z(t) = \frac{B_H(t)}{D(t)}, \quad D(t) = \max_{i=1,\dots,d}(m_i + \mu_i t).$$

Next, we state a result concerning Parisian ruin probabilities

$$\Pi(u) = \mathbb{P}\left\{\mathcal{P}_{E,[0,T_u]}(Z) > u\right\}$$

for some large class of Gaussian processes and then apply it to $\widetilde{\Pi}_{[0,S]}(N)$, rewritten in the latter form.

6.2.1 Assumptions

Let E be a compact subset of $\mathbb{R}$, containing point 0 in its interior, and let $Z(t), t \in E$ be a centered Gaussian process with a.s. continuous sample paths satisfying the following two assumptions:

Assumption A1 The variance function σ_Z of the Gaussian process Z attains its maximum on E at the unique point $\widehat{\tau} = 0$. Further, there exist positive constants $\gamma_\pm$ and $A_\pm$ such that

$$\sigma_Z(t) = 1 - A_\pm |t|^{\gamma_\pm} + o(|t|^{\gamma_\pm}) \quad \text{as} \quad t \to \pm 0. \tag{6.2}$$

Assumption A2 There exists some positive constant $\alpha \in (0, 2]$ such that

$$\mathrm{Corr}\,(Z(t), Z(s)) = 1 - D\,|t-s|^\alpha + o\,(|t-s|^\alpha) \quad \text{as } t, s \to 0.$$

Note that the majority of standard examples of continuous Gaussian processes on $[0, 1]$, such as (fractional) Brownian motion or Ornstein-Uhlenbeck process satisfy the assumptions trivially. Perhaps the simplest example of a process satisfying them in a non-trivial way is an Ornstein-Uhlenbeck process multiplied by a continuous function σ which satisfies (6.2).

Remark 6.1. *Note that it follows from **A2** that there exists such $\delta > 0$ that*

$$\mathbb{E}\left\{\left(\overline{Z}(t) - \overline{Z}(s)\right)^2\right\} < C|t-s|^\alpha$$

for all $t, s < \delta$.

As it turns out, there are two numbers

$$\nu = \min\{\alpha, \gamma_-, \gamma_+\} \quad \text{and} \quad \gamma = \max\{\gamma_-, \gamma_+\}$$

which determine *the type* of the asymptotics, but before proceeding to that, we also need the following assumption on the convergence rate of $T_u \to 0$:

Assumption B $T_u = T u^{-2/\nu}$ for some $T \in [0, \infty)$.

Next, we introduce two well-known and important constants in the theory of Gaussian extremes, see [15, 10, 1]. Define for $T \geq 0$ and $\alpha \in (0, 2]$ the generalized Pickands and Piterbarg constants

$$\mathcal{H}_\alpha^{\mathcal{P}}(T) = \lim_{\lambda\to\infty} \frac{\mathcal{H}_{\alpha,0}^{\mathcal{P}}(\lambda, T)}{\lambda} \quad \text{and} \quad \mathcal{H}_{\alpha,h}^{\mathcal{P}}(T) = \lim_{\lambda\to\infty} \mathcal{H}_{\alpha,h}^{\mathcal{P}}(T),$$

where

$$\mathcal{H}_{\alpha,h}^{\mathcal{P}}(T) = \mathbb{E}\exp\left(\sup_{t\in[-\lambda,\lambda]} \inf_{s\in[0,T]} \left(\sqrt{2}B_{\alpha/2}(t+s) - |t+s|^\alpha - h(t+s)\right)\right)$$

for such continuous h that the limit exists. We are in a position to formulate our main theorem.

Theorem 6.1. *Let $(Z(t))_{t\geq 0}$ be a centered Gaussian process satisfying assumptions **A1** and **A2**, and let T_u be a positive measurable function of u satisfying assumption **(B)**. Then*

In the Pickands case $\nu = \alpha \neq \gamma$ *we have*

$$\Pi(u) = C_S\ \mathcal{H}_\alpha^{\mathcal{P}}(D^{1/\alpha}T)\, u^{2/\nu-2/\gamma}\, \Psi(u)\Big(1+o(1)\Big),$$

with

$$C_S = A_+^{-1/\gamma_+} D^{1/\alpha}\Gamma\left(\frac{1}{\gamma_+}+1\right)1_{\gamma=\gamma_+} + A_-^{-1/\gamma_-} D^{1/\alpha}\Gamma\left(\frac{1}{\gamma_-}+1\right)1_{\gamma=\gamma_-}.$$

In the Piterbarg case $\nu = \alpha = \gamma$ *we have*

$$\Pi(u) = \mathcal{H}_{\alpha,h}^{\mathcal{P}}(D^{1/\alpha}T)\Psi(u)(1+o(1)),$$

where $h(t) = A_- D^{-1/\alpha}|t|^{\gamma_-}1_{t\leq 0} + A_+ D^{-1/\alpha}|t|^{\gamma_+}1_{t\geq 0}$.

In the Talagrand-1 case $\gamma = \nu \neq \alpha$

$$\Pi(u) = C\Psi(u)\Big(1+o(1)\Big), \quad C = \begin{cases} 1, & \gamma_+ \geq \gamma_-, \\ \exp(-\min\{A_-T^{\gamma_-}, A_+T^{\gamma_+}\}), & \gamma_+ < \gamma_-. \end{cases}$$

Now we proceed with our initial problem, to which end we first study the behaviour of $\operatorname{Var} B_H(t)/D(t)$. Note that the derivative σ' of the variance function

$$\sigma(t) = \frac{t^H}{D(t)}, \quad D(t) = \max_{i=1,\dots,d}(m_i + \mu_i t)$$

changes its sign exactly once, since

$$\sigma'(t) = \frac{t^{H-1}}{D^2(t)}G(t), \quad G(t) = HD(t) - tD'(t),$$

where $G(t)$ is monotone and decreasing, $G(0) > 0$ and $G(t) \to -\infty$ as $t \to \infty$. Since σ' must change sign (possibly in a discontinuous manner) at the optimal point t_* of σ, we have thus proved that such point is unique. Let us assume that $t_* \in (0,S)$ or $S = \infty$, since to account for the boundary maxima case an approach slightly different to ours is needed.

The maximum can either be caused by intersection of some two lines $l_\pm(t) = m_\pm + \mu_\pm t$ from $D(t)$, that is,

$$t_* = \frac{m_+ - m_-}{\mu_- - \mu_+}$$

in which case σ' is discontinuous at t_*, or by a point away from the lines' intersections, satisfying $\sigma'(t_*) = 0$, that is,

$$t_* = \frac{Hm}{\mu(1-H)}.$$

Finally, these two types of maxima can coincide, giving rise to a power-asymmetric behavior near t_*

$$\frac{\sigma(t)}{\sigma(t_*)} = 1 - A_\pm|t-t_*|^{\gamma_\pm}\Big(1+o(1)\Big)$$

with $\gamma_\pm \in \{1,2\}$. Precisely, if

$$-C_\pm^{(1)} = \frac{\sigma'_\pm(t_*)}{\sigma(t_*)} = \frac{H}{t_*} - \frac{\mu_\pm}{m_\pm + \mu_\pm t_*} < 0,$$

then $\gamma_\pm = 1$ and $A_\pm = C_\pm^{(1)}$. If on the other hand $C_\pm^{(1)} = 0$, then under the following non-degeneracy assumption

$$-C_\pm^{(2)} = \frac{\sigma''_\pm(t_*)}{\sigma(t_*)} = -\frac{H}{t_*^2} + \frac{\mu_\pm^2}{(m_\pm + \mu_\pm t_*)^2} < 0$$

we have $\gamma_\pm = 2$ and $A_\pm = C_\pm^{(2)}$.

Now we may introduce the natural asymptotic parameter

$$\widehat{N} = \frac{\sqrt{N}}{\sigma_Z(t_*)}$$

and formulate the corollary on the many inputs proportional reinsurance model.

Corollary 6.1. *Let T_N satisfy the condition*

$$\lim_{N\to\infty} T_N \widehat{N}^{1/H} = T \in [0, \infty).$$

- *If either γ_+ or γ_- equals 2, then*

$$\widetilde{\Pi}(N) = \frac{\sqrt{\pi}}{2}\frac{1}{\sqrt{A}}\Big(1_{\gamma_+=2} + 1_{\gamma_-=2}\Big)\frac{\mathcal{H}_{2H}^{\mathcal{P}}\left(T/2^{1/2H}t_*\right)}{2^{1/2H}t_*}\widehat{N}^{\zeta}\Psi(\widehat{N})\Big(1+o(1)\Big).$$

- *If both $\gamma_\pm = 1 > 2H$, then*

$$\widetilde{\Pi}(N) = \left(\frac{1}{A_-} + \frac{1}{A_+}\right)\frac{\mathcal{H}_{2H}^{\mathcal{P}}\left(T/2^{1/2H}t_*\right)}{2^{1/2H}t_*}\widehat{N}^{\zeta}\Psi(\widehat{N})\Big(1+o(1)\Big).$$

- *If both $\gamma_\pm = 1 = 2H$, then*

$$\widetilde{\Pi}(N) = \mathcal{H}_{2H,h}^{\mathcal{P}}(T/2^{1/2H}t_*)\Psi(\widehat{N})\Big(1+o(1)\Big).$$

- *If $\gamma_\pm = 1 < 2H$, then*

$$\widetilde{\Pi}(N) = \Psi(\widehat{N})\Big(1+o(1)\Big).$$

6.3 Proof of Theorem 6.1

This section is dedicated to the proof of Theorem 6.1.

6.3.1 Large vicinities.

Looking ahead, we shall prove that only a small vicinity of the optimal point contributes to the first order asymptotics, and to evaluate its contribution we shall divide this small vicinity into even smaller parts of some size $q(u)$ (referred to as the Pickands scale of the process Z, determined only by the covariance structure of Z), on which the uniform local Pickands lemma may be applied. It follows directly from the Piterbarg inequality (see, e.g. Theorem 8.1 of [11]) and the following obvious but important property of the Parisian functional:

$$\mathcal{P}_{E,F}(f) = \sup_{t\in E}\inf_{s\in F} f(t+s) \le \sup_{t\in E} f(t) \tag{6.3}$$

that

$$\limsup_{u\to\infty} \frac{\Pi_{E\setminus[-\delta_-(u),\delta_+(u)]}(u)}{u^\kappa \Psi(u)} = 0$$

for $\delta_\pm(u) = u^{-2/\gamma_\pm}\ln^{2/\gamma_\pm} u$ and all $\kappa > 0$. Since we intend to prove that $\Pi(u) \sim \mathcal{H}u^\kappa\Psi(u)$ for some $\mathcal{H}, \kappa > 0$, from this inequality will follow that $\Pi(u) \sim \Pi_{[-\delta_-(u),\delta_+(u)]}(u)$ as $u \to \infty$. We can narrow the vicinity even further by once again using (6.3) and applying Lemma 6.2 (Lemma 5.4 from [16])

$$\limsup_{u\to\infty} \frac{\Pi_{[-\delta_-(u),\delta_+(u)]\setminus[-\Lambda u^{-2/\gamma_-},\Lambda u^{-2/\gamma_+}]}(u)}{u^\kappa \Psi(u)} \le Ce^{-c\Lambda^\gamma}$$

where $\gamma = \max\{\gamma_-, \gamma_+\}$. Due to this inequality, we may concentrate on the exact asymptotics of

$$\Pi_{\Delta(u,\Lambda)}(u) \quad \text{where} \quad \Delta(u,\Lambda) = \Delta^+(u,\Lambda) \cup \Delta^-(u,\Lambda), \quad \Delta^\pm(u,\Lambda) = \pm[0, \Lambda u^{-2/\gamma_\pm}]$$

and then let $\Lambda \to \infty$.

6.3.2 Pickands intervals.

Next, we introduce the left and right Pickands intervals $\Delta_k^\pm(u,\lambda)$ with some additional parameter $\lambda > 0$

$$\Delta_k^\pm(u,\lambda) = \pm\lambda q(u)[k, k+1], \quad \text{where} \quad \nu = \min\{\alpha, \gamma_+, \gamma_-\},\ q(u) = u^{-2/\nu},$$

and the number of those fitting into the large vicinity $\Delta^\pm(u,\Lambda)$:

$$N^\pm(u,\lambda,\Lambda) = \left\lfloor \frac{|\Delta^\pm(u)|}{\lambda u^{-2/\nu}} \right\rfloor = \left\lfloor \frac{\Lambda u^{\zeta_\pm}}{\lambda} \right\rfloor, \quad \zeta_\pm = \frac{2}{\nu} - \frac{2}{\gamma_\pm} = \max\left\{\frac{2}{\alpha} - \frac{2}{\gamma_\pm}, \frac{2}{\gamma_\mp} - \frac{2}{\gamma_\pm}, 0\right\} \ge 0.$$

We shall split the proof in four cases:

Pickands case $\gamma \ne \nu = \alpha$

Piterbarg case $\gamma = \nu = \alpha$

Talagrand-1 case $\gamma = \nu \ne \alpha$

Talagrand-2 case $\gamma \ne \nu \ne \alpha$

In the Pickands case at least one of $\zeta_\pm$ is nonzero, therefore $N^\pm$ grows as $u^{\zeta_\pm}$. In both Piterbarg and Talagrand-1 cases $\zeta_+ = \zeta_- = 0$, hence $u \mapsto N^\pm$ is constant and we can set $\lambda = \Lambda$, in which case $N^\pm = 1$ and the zeroth Pickands interval coincides with the informative vicinity. The Talagrand-2 case is to be treated separately.

6.3.3 Pickands case.

To deal with the Pickands case, we employ the so-called double sum method, which is based on the Bonferroni inequality

$$\mathbf{\Sigma}_1(u,\lambda,\Lambda) - \mathbf{\Sigma}_2(u,\lambda,\Lambda) \le \Pi_{\Delta(u,\Lambda)}(u) \le \mathbf{\Sigma}_1'(u,\lambda,\Lambda),$$

where

$$\mathbf{\Sigma}_1(u,\lambda,\Lambda) = \underbrace{\sum_{k=1}^{N^+(u,\lambda,\Lambda)} \Pi_{\Delta_k^+(u,\lambda)}(u)}_{=:\mathbf{\Sigma}_1^+(u,\lambda,\Lambda)} + \underbrace{\sum_{k=1}^{N^-(u,\lambda,\Lambda)} \Pi_{\Delta_k^-(u,\lambda,\Lambda)}(u)}_{=:\mathbf{\Sigma}_1^-(u,\lambda,\Lambda)} + \underbrace{\Pi_{\Delta_0^+(u,\lambda)\cup\Delta_0^-(u,\lambda)}(u)}_{=:\mathbf{\Sigma}_0(u,\lambda)},$$

and $\mathbf{\Sigma}_1'^{\pm}$ and $\mathbf{\Sigma}_1' = \mathbf{\Sigma}_1'^{+} + \mathbf{\Sigma}_1'^{-} + \mathbf{\Sigma}_0$ denote the same $\mathbf{\Sigma}_1$ but with $N^{\pm}+1$ instead of $N^{\pm}$ (so that the collection of Pickands intervals indeed cover $\Delta(u,\Lambda)$), and finally

$$\mathbf{\Sigma}_2(u,\lambda,\Lambda) = \sum_{\substack{\kappa,\kappa'\in\{+,-\},\\ 1\le k\le N^{\kappa}(u,\lambda,\Lambda),\\ 1\le k'\le N^{\kappa'}(u,\lambda,\Lambda),\\ (\kappa,k)\neq(\kappa',k')}} \mathbb{P}\left\{\mathcal{P}_{\Delta_k^{\kappa}(u,\lambda,\Lambda),[0,T_u]}(Z) > u, \mathcal{P}_{\Delta_{k'}^{\kappa'}(u,\lambda,\Lambda),[0,T_u]}(Z) > u\right\}.$$

Since the Parisian functional is bounded from above by sup functional, we can reduce the double sum estimate to the classical (sup) case

$$\mathbf{\Sigma}_2(u,\lambda,\Lambda) \le \sum_{\substack{\kappa,\kappa'\in\{+,-\},\\ 1\le k\le N^{\kappa}(u,\lambda,\Lambda),\\ 1\le k'\le N^{\kappa'}(u,\lambda,\Lambda),\\ (\kappa,k)\neq(\kappa',k')}} \mathbb{P}\left\{\sup_{\Delta_k^{\kappa}(u,\lambda,\Lambda),} Z(t) > u, \sup_{\Delta_{k'}^{\kappa'}(u,\lambda,\Lambda),} Z(t) > u\right\},$$

and using similar arguments as in the proof of negligibility of $\Theta(u)$ in [17] (see pages 84–85) obtain

$$\lim_{\lambda\to\infty}\lim_{\Lambda\to\infty}\limsup_{u\to\infty}\frac{\mathbf{\Sigma}_2(u,\lambda,\Lambda)}{u^k\Psi(u)} = 0 \quad \text{for all } k>0. \tag{6.4}$$

We shall prove that if there exist two constants $C_{\pm}>0$ such that

$$\lim_{\lambda\to\infty}\lim_{\Lambda\to\infty}\lim_{u\to\infty}\frac{\mathbf{\Sigma}_1^{\pm}(u,\lambda,\Lambda)}{u^{\zeta_{\pm}}\Psi(u)} = C_{\pm},$$

which together with the double sum estimate above and

$$\lim_{\lambda\to\infty}\lim_{u\to\infty}\frac{\mathbf{\Sigma}_0(u,\lambda)}{\Psi(u)} = \mathcal{H}_0 \in (0,\infty), \tag{6.5}$$

yields

$$\lim_{\Lambda\to\infty}\lim_{u\to\infty}\frac{\Pi_{\Delta(u,\Lambda)}(u)}{u^{\zeta}\Psi(u)} = C,$$

where $\zeta = \max\{\zeta_+,\zeta_-\}>0$ and $C = C_+1_{\zeta=\zeta_+} + C_-1_{\zeta=\zeta_-}$. Finally, we obtain

$$\Pi(u) \sim Cu^{\zeta}\Psi(u).$$

6.3.4 Piterbarg and Talagrand-1 cases.

As noted before, in both Piterbarg and Talagrand-1 cases $u \mapsto N^{\pm}(u,\lambda,\Lambda)$ is constant. We may set $\lambda=\Lambda$, which makes this constant equal one, and therefore

$$\Pi_{\Delta(u,\Lambda)} = \mathbf{\Sigma}_0(u,\Lambda).$$

By (6.5), we have

$$\lim_{\Lambda\to\infty}\lim_{u\to\infty}\frac{\mathbf{\Sigma}_0(u,\Lambda)}{\Psi(u)} = \mathcal{H}_0$$

for some $\mathcal{H}_0>0$, which together with Lemma 6.4.3 yields $\Pi(u)\sim\mathcal{H}_0\Psi(u)$. In the Talagrand case $\nu\neq\alpha$ we shall see that $\mathcal{H}_0=1$, and therefore $\Pi(u)\sim\Psi(u)$.

6.3.5 Talagrand-2 case.

In the Talagrand-2 case we shall directly (that is, without appealing to Pickands intervals) prove that there exists positive and finite limit

$$\lim_{u\to\infty} \frac{\Pi_{\Delta(\lambda,u)}(u)}{\Psi(u)} = \begin{cases} 1, & \gamma_- < \gamma_+, \\ \exp(-\min\{A_- T^{\gamma_-}, A_+ T^{\gamma_+}\}), & \gamma_+ < \gamma_-, \end{cases}$$

which ceases to depend on λ as long as $\lambda > T$.

6.3.6 Asymptotics of $\Sigma_0(u,\lambda)$ and the Piterbarg and Talagrand-1 cases

Denote

$$q(u) = u^{-2/\nu}, \quad \nu = \min\{\alpha, \gamma_+, \gamma_-\}.$$

Note that

$$u^{2/\alpha} q(u) \xrightarrow[u\to\infty]{} 1_{\nu=\alpha} \quad \text{and} \quad u^{2/\gamma_\pm} q(u) \xrightarrow[u\to\infty]{} 1_{\nu=\gamma_\pm}.$$

To apply the uniform local Pickands Lemma 6.1, let us rewrite the probability $\mathbf{\Sigma}_0(u,\lambda)$ in terms of a standardized process as follows:

$$\mathbf{\Sigma}_0(u,\lambda) = \mathbb{P}\left\{\mathcal{P}_{\Delta_0^+(u,\lambda)\cup\Delta_0^-(u,\lambda)}(Z) > u\right\} = \mathbb{P}\left\{\mathcal{P}'(\xi_{u,0}) > u\right\},$$

where we have defined $\mathcal{P}' = \mathcal{P}_{[-\lambda,\lambda],[0,1]}$ and the family $\{\xi_{u,0} \colon u > 0\}$ of centered Gaussian processes by

$$\xi_{u,0}(t,s) = \frac{\overline{Z}(q(u)t + T_u s)}{1 + g(q(u)t + T_u s)}, \quad 1 + g(t) = \frac{1}{\sigma_Z(t)}.$$

Note that in contrast to $\xi_{u,k}$, $k > 0$ (see below), the process $\xi_{u,0}$ is defined for $t \in [-\lambda, \lambda]$, not $[0, \lambda]$. This is neither a coincidence, nor a technical decision: the two adjacent intervals near the optimal point cannot be treated separately as it will be evident from the result.

By (6.2) and the definition of g we have

$$u^2 g(q(u)t + T_u s) \to h(t + Ts) = h^+(t + Ts)1_{\nu=\gamma_+} + h^-(t + Ts)1_{\nu=\gamma_-},$$

where

$$h^\pm(\mu) = A_\pm |\mu|^{\gamma_\pm} 1_{\pm\mu>0}.$$

By assumption **A2** we have

$$u^2 \mathbb{E}\left\{\left|Z(q(u)t + T_u s) - Z(q(u)t' + T_u s')\right|^2\right\} \to 2D1_{\nu=\alpha}\left|(t - t') + T(s - s')\right|^\alpha,$$

which means that the condition **C2** is satisfied with

$$\eta(t,s) = B_{\alpha/2}(t + s)1_{\nu=\alpha}, \quad (t,s) \in [-D^{1/\alpha}\lambda, D^{1/\alpha}\lambda] \times [0, D^{1/\alpha}T]$$

for $T \geq 0$. By the uniform local Pickands Lemma 6.1 (condition (C3) is obviously satisfied) we have

$$\lim_{u\to\infty} \frac{\mathbf{\Sigma}_0(u,\lambda)}{\Psi(u)} = \mathcal{H}_{\eta,h}^{\mathcal{P}_0}\Big([-D^{1/\alpha}\lambda, D^{1/\alpha}\lambda] \times [0, D^{1/\alpha}T]\Big),$$

where $\mathcal{H}_{\eta,h}^{\mathcal{P}_0}(E) = \mathbb{E}\left\{e^{\mathcal{P}(\eta^h)}\right\}$, $\mathcal{P}_0 = \mathcal{P}_{[-\lambda,\lambda],[0,T]}$ and

$$\eta^h(t,s) = \Big(\sqrt{2}B_{\alpha/2}(t+s) - |t+s|^\alpha\Big)1_{\nu=\alpha} - A_- D^{-\gamma_-/\alpha}|t+s|^{\gamma_-}1_{t+s\leq 0,\nu=\gamma_-} - A_+ D^{-\gamma_+/\alpha}|t+s|^{\gamma_+}1_{t+s\geq 0,\nu=\gamma_+}. \quad (6.6)$$

Piterbarg case.

To prove the main theorem in the Piterbarg case $\gamma = \alpha = \nu$, that is when η^h has all the terms

$$\eta^h(t,s) = \left(\sqrt{2}B_{\alpha/2}(t+s) - |t+s|^\alpha\right) - A_- D^{-\gamma_-/\alpha}|t+s|^{\gamma_-}1_{t+s\le 0} - A_+ D^{-\gamma_+/\alpha}|t+s|^{\gamma_+}1_{t+s\ge 0},$$

it remains to apply the standard result on the existence of Piterbarg constants to see that

$$\lim_{\lambda\to\infty}\lim_{u\to\infty}\frac{\mathbf{\Sigma}_0(u,\lambda)}{\Psi(u)} = \lim_{\lambda\to\infty}\mathcal{H}^{\mathcal{P}_0}_{\eta,h}\Big([-D^{1/\alpha}\lambda, D^{1/\alpha}\lambda]\times[0, D^{1/\alpha}T]\Big) = \mathcal{H}^{\mathcal{P}_0}_{\eta,h}(D^{1/\alpha}T)$$

exists and is finite. This ends the proof of the main theorem in the Piterbarg case.

Talagrand-1 case.

In the Talagrand-1 case $\gamma = \nu \neq \alpha$ and, therefore, the random part disappears from (6.6), whereas all non-random terms are present:

$$\eta^h(t,s) = -A_+ D^{-\gamma_+/\alpha}|t+s|^{\gamma_+}1_{t+s\ge 0} - A_- D^{-\gamma_-/\alpha}|t+s|^{\gamma_-}1_{t+s\le 0}$$

It remains to calculate $\mathcal{P}_0(\eta^h)$ explicitly: if $\lambda > T$, we have

$$\begin{aligned}
\mathcal{P}_0\left(\eta^h\right) &= \sup_{t\in[-\lambda,\lambda]}\inf_{s\in[0,T]}\Big(-A_+|t+s|^{\gamma_+}1_{t+s\ge 0} - A_-|t+s|^{\gamma_-}1_{t+s\le 0}\Big)\\
&= \max\left\{\sup_{t\in[-\lambda,-T]}\inf_{s\in[0,T]}(-A_-|t+s|^{\gamma_-}),\ \sup_{t\in[-T,\lambda]}\inf_{s\in[0,T]}(-A_+|t+s|^{\gamma_+})\right\}\\
&= \max\left\{\sup_{t\in[-\lambda,-T]}(-A_-|t|^{\gamma_-}),\ \sup_{t\in[-T,\lambda]}(-A_+|t+T|^{\gamma_+})\right\}\\
&= \max\{-A_-T^{\gamma_-}, 0\} = 0.
\end{aligned}$$

Therefore, by lemma above in the Talagrand case

$$\mathcal{H}^{\mathcal{P}_0}_{\eta,h}\Big([-D^{1/\alpha}\lambda, D^{1/\alpha}\lambda]\times[0, D^{1/\alpha}T]\Big) = 1$$

for all $\lambda > T$. Thus, we have proved that

$$\lim_{\lambda\to\infty}\lim_{u\to\infty}\frac{\mathbf{\Sigma}_0(u,\lambda)}{\Psi(u)} = 1.$$

This ends the proof of the main theorem in the Talagrand case.

6.3.7 Talagrand-2 case.

To apply the uniform local Pickands Lemma 6.1, let us rewrite the probability $\Pi_{\Delta(u,\Lambda)}(u)$ in terms of a standardized process as follows. First, observe that the trivial equality

$$t+s = (t+s)1_{t+s\ge 0} + (t+s)1_{t+s\le 0}, \quad (t,s)\in[-\lambda u^{-2/\gamma_-}, \lambda u^{-2/\gamma_+}]\times[0,T_u]$$

may be rewritten as

$$t+s = q(u,t',s') = q_+(u,t',s') + q_-(u,t',s'), \quad (t',s')\in[-\lambda,\lambda]\times[0,1].$$

where
$$q_{\pm}(u,t',s') = \left(u^{-2/\gamma_{\pm}}t' + T_u s'\right)1_{\pm(u^{-2/\gamma_{\pm}}t'+T_u s')\geq 0}.$$

Using this reparametrization, we rewrite

$$\Pi_{\Delta(u,\lambda)}(u) = \mathbb{P}\left\{\mathcal{P}_{[-\lambda u^{-2/\gamma_-},\lambda u^{-2/\gamma_+}]}(Z) > u\right\} = \mathbb{P}\left\{\mathcal{P}'(\xi_{u,0}) > u\right\},$$

where we have defined $\mathcal{P}' = \mathcal{P}_{[-\lambda,\lambda],[0,1]}$ and the family $\{\xi_{u,0}\colon u > 0\}$ of centered Gaussian processes by

$$\xi_{u,0}(t',s') = \frac{\overline{Z}(q(u,t',s'))}{1+g(q(u,t',s'))}, \quad 1+g(t) = \frac{1}{\sigma_Z(t)}.$$

Note that
$$u^{2/\gamma_{\pm}}q_{\pm}(u,t',s') \to \left(t' + 1_{\nu=\gamma_{\pm}}Ts'\right)1_{\pm(t'+1_{\nu=\gamma_{\pm}}Ts')\geq 0}$$

uniformly in (t',s'). By (6.2) and the definition of g we have

$$\begin{aligned}
u^2 g(q(u,t',s')) &\sim A_+|u^{2/\gamma_+}q_+(u,t',s')|^{\gamma_+} + A_-|u^{2/\gamma_-}q_-(u,t',s')|^{\gamma_-} \\
&\sim A_+|t' + 1_{\nu=\gamma_+}Ts'|^{\gamma_+}1_{t'+1_{\nu=\gamma_+}Ts'\geq 0} + A_-|t'+1_{\nu=\gamma_-}Ts'|^{\gamma_-}1_{t'+1_{\nu=\gamma_-}Ts'\geq 0}.
\end{aligned}$$

By assumption **A2** we have

$$u^2\mathbb{E}\left\{\left|Z(q(u,t_1',s_1')) - Z(q(u,t_2',s_2'))\right|^2\right\} \to 0,$$

which means that the condition **C2** $\eta(t,s) = 0$.

By the uniform local Pickands Lemma 6.1 (condition **C3** is obviously satisfied) we have

$$\lim_{u\to\infty}\frac{\Pi_{\Delta(u,\lambda)}(u)}{\Psi(u)} = \mathcal{H}_{0,h}^{\mathcal{P}_0}([-\lambda,\lambda]\times[0,T]),$$

where $\mathcal{H}_{0,h}^{\mathcal{P}_0}(E) = e^{\mathcal{P}_0(-h)}$, $\mathcal{P}_0 = \mathcal{P}_{[-\lambda,\lambda],[0,T]}$ and

$$h(t,s) = A_-|t+1_{\nu=\gamma_-}s|^{\gamma_-}1_{t+1_{\nu=\gamma_-}s\leq 0} + A_+|t+1_{\nu=\gamma_+}s|^{\gamma_+}1_{t+1_{\nu=\gamma_+}s\geq 0}.$$

Since we are looking at a case where $\gamma \neq \nu$, either $1_{\nu=\gamma_-}$ or $1_{\nu=\gamma_+}$ is zero. Suppose, $\nu = \gamma_- \neq \gamma_+$. Then
$$h(t,s) = -A_-|t+s|^{\gamma_-}1_{t+s\leq 0} - A_+|t|^{\gamma_+}1_{t\geq 0}.$$

therefore, we have

$$\begin{aligned}
\mathcal{P}(h) = \max&\left\{\sup_{t\in[-\lambda,0]}\inf_{s\in[0,t]}\big(-a_-|t+s|^{\gamma_-}1_{t+s\leq 0}\big), \sup_{t\in[0,\lambda]}\inf_{s\in[0,t]}\big(-a_+|t|^{\gamma_+}\big)\right\} \\
&= \max\left\{\sup_{t\in[-\lambda,0]}\inf_{\mu\in[t,t+t]}\big(-a_-|\mu|^{\gamma_-}1_{\mu\leq 0}\big), 0\right\} = 0
\end{aligned}$$

since the first term is non-positive.

If, on the other hand, $\nu = \gamma_+ \neq \gamma_-$, we have

$$h(t,s) = -a_-|t|^{\gamma_-}1_{t\leq 0} - a_+|t+s|^{\gamma_+}1_{t+s\geq 0}$$

and therefore for $\lambda > T$ we have

$$\mathcal{P}(h) = \max\left\{\sup_{t\in[-\lambda,0]}\inf_{s\in[0,T]}\Big(-A_-|t|^{\gamma_-} - A_+|t+s|^{\gamma_+}1_{t+s\geq 0}\Big), \sup_{t\in[0,\lambda]}\inf_{s\in[0,T]}\Big(-A_+|t+s|^{\gamma_+}\Big)\right\}$$

$$= \max\left\{\sup_{t\in[-\lambda,0]}\inf_{\mu\in[t,t+T]}\Big(-A_-|t|^{\gamma_-} - A_+|\mu|^{\gamma_+}1_{\mu\geq 0}\Big), -A_+|T|^{\gamma_+}\right\}$$

$$= \max\left\{\sup_{t\in[-\lambda,0]}\min\left\{-A_-|t|^{\gamma_-}, \inf_{\mu\in[0,\max(t+T,0)]}\Big(-A_-|t|^{\gamma_-} - A_+|\mu|^{\gamma_+}\Big)\right\}, -A_+|T|^{\gamma_+}\right\}$$

$$= \max\left\{\sup_{t\in[-\lambda,0]}\Big(-A_-|t|^{\gamma_-} - A_+|\max(t+T,0)|^{\gamma_+}\Big), -A_+|T|^{\gamma_+}\right\}$$

$$= \max\left\{\sup_{t\in[-\lambda,-T]}\Big(-A_-|t|^{\gamma_-}\Big), \sup_{t\in[-T,0]}\Big(-A_-|t|^{\gamma_-} - A_+|t+T|^{\gamma_+}\Big), -A_+|T|^{\gamma_+}\right\}$$

$$= \max\left\{-A_-|T|^{\gamma_-}, \sup_{\mu\in[0,T]}\Big(-A_-|\mu|^{\gamma_-} - A_+|T-\mu|^{\gamma_+}\Big), -A_+|T|^{\gamma_+}\right\}$$

$$= -\min\left\{A_-T^{\gamma_-}, A_+T^{\gamma_+}\right\}$$

We have thus proved that

$$\mathcal{H}^{\mathcal{P}}_{0,h}([-\lambda,\lambda]\times[0,T]) = e^{\mathcal{P}(-h)} = \begin{cases} 1, & \gamma_- < \gamma_+, \\ \exp(-\min\{A_-T^{\gamma_-}, A_+T^{\gamma_+}\}), & \gamma_+ < \gamma_- \end{cases}$$

for all $\lambda > T$.

6.3.8 Pickands case.

Now we proceed to the Pickands case.

To find the aforementioned asymptotics of $\mathbf{\Sigma}^{\pm}_1(u,\lambda,\Lambda)$ we shall first find the uniform in $k \in \{1,\dots,N^{\pm}(u,\lambda,\Lambda)\}$ asymptotics of each summand $\Pi_{\Delta^{\pm}(u,\lambda,\Lambda)}(u)$ and then sum them up. To this end, let us rewrite the probability in the form required for the uniform local Pickands Lemma 6.1

$$\Pi_{\Delta^{\pm}_k(u,\lambda,\Lambda)}(u) = \mathbb{P}\left\{\mathcal{P}_{\Delta^{\pm}_k(u,\lambda,\Lambda),[0,T_u]}(Z) > u\right\} = \mathbb{P}\left\{\mathcal{P}''(\xi^{\pm}_{u,k}) > u\right\},$$

where we have defined $\mathcal{P}'' = \mathcal{P}_{[0,\lambda],[0,1]}$ and the family

$$\{\xi^{\kappa}_{u,k} \colon \kappa \in \{+,-\}, 1 \leq k \leq N^{\pm}(u,\lambda,\Lambda), u > 0\}$$

of centered Gaussian processes by

$$\xi^{\pm}_{u,k}(t,s) = \frac{\widetilde{Z}^{\pm}_{u,k}(t,s)}{1+h^{\pm}_{u,k}(t,s)}, \quad 1+h^{\pm}_{u,k}(t,s) = \frac{1}{\sigma_Z(\pm q(u)(\lambda k+t)+T_u s)}$$

and

$$\widetilde{Z}^{\pm}_{u,k}(t,s) = \frac{Z(\pm q(u)(\lambda k+t)+T_u s)}{\sigma_Z(\pm q(u)(\lambda k+t)+T_u s)},$$

$t \in [0, \lambda], s \in [0, 1]$. Note that $\widetilde{Z}^{\pm}_{u,k}$ is a centered Gaussian random field with unit variance and continuous paths. Besides, $h^{\pm}_{u,k} \in C_0([0, \lambda] \times [0, T])$, that is, it is a continuous function on $[0, \lambda] \times [0, T]$, such that $h^{\pm}_{u,k}(0, 0) = 0$.

There is, however, a pitfall in trying to apply uniform local Pickands Lemma 6.1 directly to $\xi^{\pm}_{u,k}$. Even though $u^2 h_{u,k}$ may have a limit h for each k, it is never uniform. In other words, the condition **(C1)**

$$\lim_{u\to\infty} \sup_{k\in K_u, (t,s)\in[0,\lambda]\times[0,T]} \left| u^2 h_{u,k}(t, s) - h(t, s) \right| = 0$$

is not satisfied. To get around this inconvenience, we shall coarsen the inequality describing the event by taking $1 + h_{u,k}$ out of $\mathcal{P}''$

$$\mathbb{P}\left\{\mathcal{P}''(\widetilde{Z}_{u,k})/U(1 + h^{\pm}_{u,k}) > u\right\} \le \Pi_{\Delta^{\pm}_k(u,\lambda)}(u)$$
$$= \mathbb{P}\left\{\mathcal{P}''(\xi_{u,k}) > u\right\} \le \mathbb{P}\left\{\mathcal{P}''(\widetilde{Z}_{u,k})/L(1 + h^{\pm}_{u,k}) > u\right\},$$

where

$$U(f) = \sup_{t\in[0,\lambda]} \sup_{s\in[0,T]} f(t + s) \quad \text{and} \quad L(f) = \inf_{t\in[0,\lambda]} \inf_{s\in[0,T]} f(t + s).$$

Let us rewrite it in a handier fashion as

$$\mathbb{P}\left\{\mathcal{P}''(\widetilde{Z}_{u,k}) > u\Big(1 + U(h^{\pm}_{u,k})\Big)\right\} \le \Pi_{\Delta^{\pm}_k(u,\lambda)}$$
$$= \mathbb{P}\left\{\mathcal{P}''(\xi_{u,k}) > u\right\} \le \mathbb{P}\left\{\mathcal{P}''(\widetilde{Z}_{u,k}) > u\Big(1 + L(h^{\pm}_{u,k})\Big)\right\}.$$

Using the assumption that $A_{\pm} > 0$, we get

$$L(h^{+}_{u,k}) = A_{+}\Big|\lambda q(u)k\Big|^{\gamma_+}, \quad U(h^{+}_{u,k}) = A_{+}\Big|\lambda q(u)(k + 1) + T_u\Big|^{\gamma_+}$$

and

$$L(h^{-}_{u,k}) = A_{-}\Big| - \lambda q(u)k + T_u\Big|^{\gamma_-}, \quad U(h^{-}_{u,k}) = A_{-}\Big|\lambda q(u)(k + 1)\Big|^{\gamma_-}.$$

All four bounds can be rewritten as follows:

$$L(h^{\pm}_{u,k}) = A_{\pm}\Big|\lambda q(u)k - q(u)m_{\pm}(u)\Big|^{\gamma_\pm}, \quad U(h^{\pm}_{u,k}) = A_{\pm}\Big|\lambda q(u)k + q(u)(\lambda + m_{\mp}(u))\Big|^{\gamma_\pm},$$

where $m_+(u) = 0$ and $m_-(u) = T_u/q(u) \to T$ as $u \to \infty$. It is important for us that $m_\pm$ do not depend on k and have finite limits as $u \to \infty$.

In order to apply the uniform local Pickands Lemma 6.1 to the upper bound, we set

$$g_{u,k}(\lambda) = u\Big(1 + A_{\pm}\Big|\lambda q(u)k - q(u)m_{\pm}(u)\Big|^{\gamma_\pm}\Big)$$

and note that the condition **(C2)** remains valid with $g_{u,k}$ instead of u. It now follows directly from the uniform local Pickands Lemma 6.1 that with $\mathcal{P} = \mathcal{P}_{[0,\lambda],[0,T]}$ we have

$$\mathbb{P}\left\{\mathcal{P}''(\widetilde{Z}_{u,k}) > g_{u,k}(\lambda)\right\} = \mathcal{H}^{\mathcal{P}}_{2H}\left(D^{1/\alpha}\lambda, D^{1/\alpha}T\right)\Psi(u)$$
$$\times \exp\left(-A_{\pm}\Big|\lambda u^{-\zeta_\pm}k - u^{-\zeta_\pm}m_{\pm}(u)\Big|^{\gamma_+}\right)(1 + o(1)),$$

where $o(1)$ is uniform in $k \in \{1, \dots, N^{\pm}(u, \lambda, \Lambda)\}$. Thus,

$$\mathbf{\Sigma}_1^{\pm}(u, \lambda, \Lambda) \le \sum_{k=1}^{N^{\pm}(u,\lambda,\Lambda)} \mathbb{P}\left\{\mathcal{P}(\widetilde{Z}_{u,k}) > g_{u,k}(\lambda)\right\}$$

$$\sim \mathcal{H}_{2H}^{\mathcal{P}}\left(D^{1/\alpha}\lambda, D^{1/\alpha}T\right)\Psi(u) \sum_{k=1}^{N^{\pm}(u,\lambda,\Lambda)} \exp\left(-A_{\pm}\left|\lambda u^{-\zeta_{\pm}}k - u^{-\zeta_{\pm}}m_{\pm}(u)\right|^{\gamma_{\pm}}\right)$$

$$\sim \frac{\mathcal{H}_{2H}^{\mathcal{P}}\left(D^{1/\alpha}\lambda, D^{1/\alpha}T\right)}{\lambda} u^{\zeta_{\pm}}\Psi(u) \int_0^{\Lambda} e^{-A_{\pm}x^{\gamma_{\pm}}}\, dx$$

and, letting $u \to \infty$, then $\Lambda \to \infty$ and finally $\lambda \to \infty$, we see that for large enough u, holds

$$\lim_{\lambda\to\infty}\lim_{\Lambda\to\infty}\limsup_{u\to\infty}\frac{\mathbf{\Sigma}_1^{\pm}(u,\lambda,\Lambda)}{u^{\zeta_{\pm}}\Psi(u)} \le \Gamma\left(\frac{1}{\gamma_{\pm}}+1\right)A_{\pm}^{-1/\gamma_{\pm}}D^{1/\alpha}\mathcal{H}_{\alpha}^{\mathcal{P}}(D^{1/\alpha}T)$$

where

$$\mathcal{H}_{\alpha}^{\mathcal{P}}(T) = \lim_{\lambda\to\infty}\frac{\mathcal{H}_{\alpha}^{\mathcal{P}}(\lambda, T)}{\lambda} \in (0,\infty).$$

Using lower bound in much the same fashion, we obtain

$$\lim_{\lambda\to\infty}\lim_{\Lambda\to\infty}\limsup_{u\to\infty}\frac{\mathbf{\Sigma}_1^{\pm}(u,\lambda,\Lambda)}{u^{\zeta_{\pm}}\Psi(u)} = \Gamma\left(\frac{1}{\gamma_{\pm}}+1\right)A^{-1/\gamma_{\pm}}D^{1/\alpha}\mathcal{H}_{\alpha}^{\mathcal{P}}(D^{1/\alpha}T).$$

The same formula obviously holds for $\mathbf{\Sigma}_1'^{\pm}$. To conclude the proof in the Pickands case, it remains to notice that

$$\begin{aligned}
&\lim_{\lambda\to\infty}\lim_{\Lambda\to\infty}\lim_{u\to\infty}\frac{\mathbf{\Sigma}_1(u,\lambda,\Lambda)}{u^{\zeta}\Psi(u)} \\
&\quad= \lim_{\lambda\to\infty}\lim_{\Lambda\to\infty}\lim_{u\to\infty}\left(u^{\zeta_+-\zeta}\frac{\mathbf{\Sigma}_1^{+}(u,\lambda,\Lambda)}{u^{\zeta_+}\Psi(u)} + u^{\zeta_--\zeta}\frac{\mathbf{\Sigma}_1^{-}(u,\lambda,\Lambda)}{u^{\zeta_-}\Psi(u)} + u^{-\zeta}\frac{\mathbf{\Sigma}_0(u,\lambda)}{\Psi(u)}\right) \\
&\quad= \left(\Gamma\left(\frac{1}{\gamma_+}+1\right)A_+^{-1/\gamma_+}1_{\zeta=\zeta_+} + \Gamma\left(\frac{1}{\gamma_-}+1\right)A_-^{-1/\gamma_-}1_{\zeta=\zeta_+}\right)D^{1/\alpha}\mathcal{H}_{\alpha}^{\mathcal{P}}(D^{1/\alpha}T),
\end{aligned}$$

that the same is obviously true for $\mathbf{\Sigma}_1'$, and that $1_{\zeta=\zeta_{\pm}} = 1_{\gamma=\gamma_{\pm}}$.

6.4 Appendix

In this appendix, we recall some known results necessary for the proofs of previous section.

6.4.1 Parisian functional continuity

We show now that the Parisian functional $\mathcal{P}\colon C(E\times F) \to \mathbb{R}$ is continuous in uniform topology. To this end, we take an arbitrary function $f \in C(E\times F)$ and a family $\{f_{\varepsilon}, \varepsilon > 0\} \subset C(E\times F)$, which converges to a function $f \in C(E\times F)$ uniformly

$$\sup_{(t,s)\in E\times F}\Big|f(t,s) - f_{\varepsilon}(t,s)\Big| < \varepsilon$$

as $\varepsilon \to 0$, from which we obtain

$$\begin{cases} f(t,s) - f_\varepsilon(t,s) < \varepsilon, \\ f_\varepsilon(t,s) - f(t,s) < \varepsilon \end{cases} \qquad \text{for all} \quad (t,s) \in E \times F.$$

Hence,

$$\sup_{t\in E} \inf_{s\in F} f(t,s) < \varepsilon + \sup_{t\in E} \inf_{s\in F} f_\varepsilon(t,s) \quad \text{and} \quad \sup_{t\in E} \inf_{s\in F} f_\varepsilon(t,s) < \varepsilon + \sup_{t\in E} \inf_{s\in F} f(t,s),$$

or, equivalently,

$$\left| \sup_{t\in E} \inf_{s\in F} f(t,s) - \sup_{t\in E} \inf_{s\in F} f_\varepsilon(t,s) \right| = \left| \mathcal{P}(f) - \mathcal{P}(f_\varepsilon) \right| < \varepsilon.$$

6.4.2 Uniform local Pickands lemma

The following lemma is from [15], it is reproduced here for the reader's convenience. Let

$$\xi_{u,\tau_u}(t) = \frac{Z_{u,\tau_u}(t)}{1 + h_{u,\tau_u}(t)}, \quad t \in E, \tau_u \in K_u,$$

be a family of centered Gaussian random fields with Z_{u,τ_u} a centered Gaussian random field with unit variance and continuous paths, and h_{u,τ_u} belonging to $C_0(E)$, that is, h_{u,τ_u} is a continuous function on E, such that $h_{u,\tau_u}(0) = 0$. We assume that E is a compact subset of $\mathbb{R}^d$ and $0 \in E$. The Parisian functional $\Gamma_{E,F}$

$$\Gamma_{E,F}(X) = \sup_{t\in E} \inf_{s\in F} X(t+s)$$

satisfies the conditions

(F1) there exists $c > 0$ such that $\Gamma(f) \le c \sup_{t\in E} f(t)$ for any $f \in C(E)$

(F2) $\Gamma(af + b) = a\Gamma(f) + b$ for any $f \in C(E)$ and $a > 0$, $b \in \mathbb{R}$

of the paper [15]. Therefore, under conditions **(C0)**–**(C3)**

(C0) $\lim_{u\to\infty} \inf_{\tau_u \in K_u} g_{u,\tau_u} = \infty$

(C1) there exists $h \in C_0(E)$ such that

$$\lim_{u\to\infty} \sup_{\tau_u \in K_u, t\in E} \left| g^2_{u,\tau_u} h_{u,\tau_u}(t) - h(t) \right| = 0$$

(C2) there exists $\theta_{u,\tau_u}(t,s)$ such that

$$\lim_{u\to\infty} \sup_{\tau_u \in K_u} \sup_{s \ne t \in E} \left| g^2_{u,\tau_u} \frac{\mathbb{E}\left\{ |Z_{u,\tau_u}(t) - Z_{u,\tau_u}(s)|^2 \right\}}{2\theta_{u,\tau_u}(t,s)} - 1 \right| = 0,$$

where for some centered Gaussian random field $\eta(t)$, $t \in \mathbb{R}^d$ with continuous paths and $\eta(0) = 0$,

$$\lim_{u\to\infty} \sup_{\tau_u \in K_u} \left| \theta_{u,\tau_u}(t,s) - \mathbb{E}\left\{ |\eta(t) - \eta(s)|^2 \right\} \right| = 0.$$

(C3) there exists $a > 0$ such that

$$\limsup_{u\to\infty} \sup_{\tau_u\in K_u} \sup_{s\neq t\in E} \frac{\theta_{u,\tau_u}(t,s)}{\sum_{i=1}^d |s_i - t_i|^a} < \infty,$$

$$\lim_{\varepsilon\to 0} \limsup_{u\to\infty} \sup_{\tau_u\in K_u} \sup_{\|t-s\|<\varepsilon, t,s\in E} g_{u,\tau_u}^2 \mathbb{E}\left\{\left[Z_{u,\tau_u}(t) - Z_{u,\tau_u}(s)\right] Z_{u,\tau_u}(0)\right\} = 0.$$

we have

Lemma 6.1. *Under assumptions **(C0)**–**(C3)** and if $\mathbb{P}(\Gamma_{E,F}(\xi_{u,\tau_u}) > g_{u,\tau_u}) > 0$ for all $\tau_u \in K_u$ and all large u, then*

$$\lim_{u\to\infty} \sup_{\tau_u\in K_u} \left| \frac{\mathbb{P}\{\Gamma(\xi_{u,\tau_u}) > g_{u,\tau_u}\}}{\Psi(g_{u,\tau_u})} - \mathcal{H}_{\eta,h}^{\Gamma}(E) \right| = 0,$$

where

$$\mathcal{H}_{\eta,h}^{G}(E) = \mathbb{E}\left\{e^{\Gamma(\eta^h)}\right\}, \quad \eta^h(t) = \sqrt{2}\eta(t) - \sigma_\eta^2(t) - h(t).$$

6.4.3 Large vicinity cut-off lemma

Next lemma is from [16] (Lemma 5.4), but instead of the version therefrom, we give a version suitable for our needs. This lemma allows one to get rid of the complement of the Piterbarg vicinity in all three (Piterbarg, Pickands and Talagrand) cases (see proof of Theorem 6.1).

Lemma 6.2. *There exist positive constants C, c, u_0 and Λ_0 such that for $\Lambda \geq \Lambda_0$ and $u \geq u_0$*

$$\mathbb{P}\left\{\exists t \in \left[-\delta_-(u), \Lambda u^{-2/\gamma_-}\right] \cup \left[\Lambda u^{-2/\gamma_+}, \delta_+(u)\right] : Z(t) > u\right\} \leq Ce^{-c\Lambda^\gamma} \mathbb{P}\left\{Z(0) > u\right\}.$$

where $\delta_\pm(u) = u^{-2/\gamma_\pm} \ln^{2/\gamma_\pm} u$.

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